P2 – Group 4[†]

Report

L'Hôpital's Rule: Origins

A line-by-line commentary of Chapter 9 of L'Hôpital's *Analyse des infinitements petits, pour l'intelligence des lignes courbes* (1696) and a letter from Bernoulli to L'Hôpital from July 1694.

Victor Elgersma
v.j.b.elgersma@students.uu.nl

Vera Belde
v.m.p.belde@students.uu.nl

Submitted: 7 April, 2026

[†]This report constitutes the written component of P2 – Group 4. For the oral presentation materials, see [1].

Contents

1. Introduction

L'Hôpital's rule is a well-known result[2] in calculus that allows one to find the value of a quotient of two functions $\frac{f(x)}{g(x)}$ at points where both the numerator and denominator are zero . It first appeared in print in Chapter 9 of the Marquis de L'Hôpital's *Analyse des infinitements petits, pour l'intelligence des lignes courbes* in 1696 [3]

It is known that L'Hôpital received private lessons from Bernoulli, and L'Hôpital credits him generously (but vaguely) in his preface: "I have made plain use of [Johann Bernoulli's] discoveries and those of Mr Leibniz" [4]. So, how much of the *Analyse* was L'Hôpital's original work? In particular, who came up with L'Hôpital's rule?

Bernoulli's letters to L'Hôpital were rediscovered by Gustav Eneström in 1879, revealing that L'Hôpital paid Bernoulli for priority access to his discoveries, as well as the promise not to publish them or disclose them to anyone else, in what Clifford Truesdell has called one of the most 'most unusual arrangements in the history of science' [5].

This report provides a line-by-line commentary of Chapter 9 of the *Analyse* and a letter dated July 22nd, 1693 from Bernoulli to L'Hôpital [6, L28]. We show that the primary arguments in Chapter 9 of the *Analyse* are virtually identical to those found in Bernoulli's letter. The sketches are nearly identical, except that L'Hôpital approaches the limit from above whereas Bernoulli approaches it from below. The examples used are also marginally different. The most important difference for the reader lies in L'Hôpital's explicit invocation of the First Postulate of Differential Calculus — a step which Bernoulli left implicit. This distinction is consistent with their respective roles: as a textbook author, L'Hôpital was expected to explain every step clearly, whereas Bernoulli could safely omit self-evident inferences in his private letter to a seasoned student.

[2]For a modern statement and proof, see e.g. [2]

2. Historical Background

2.1. Dramatis Personae

Johann Bernoulli (1667-1748)

Le Marquis de L'Hôpital (1661–1704)

The Tutor and the Student: Bernoulli (left) and L'Hôpital (right).

2.2. Timeline

1691	Johann Bernoulli arrives in Paris; begins tutoring the Marquis de L'Hôpital in the "new calculus" of Leibniz.
1692	Bernoulli leaves Paris for Basel but continues to provide lessons via correspondence.
1694	L'Hôpital offers Bernoulli a 300-livre annual retainer in exchange for exclusive access to his mathematical discoveries.
July 1694	Letter 28: Bernoulli sends the solution to the indeterminate form 0/0 to L'Hôpital as part of their agreement.
1696	L'Hôpital publishes the *Analyse des infinitements petits* anonymously. It contains the rule but vaguely credits Johann Bernoulli, who at this point has obtained a Professorship in Groningen.
1704	The Marquis dies; Bernoulli begins to publicly claim authorship of the methods in the *Analyse*.

2.3. The significance of the *Analyse* (1696)

Published in 1696 in Paris, the *Analyse* (1696) is the first known textbook on differential calculus. It was well-known and closely read throughout the 17th century, serving as the first introduction to the subject to many french mathematicians. It is the reason that L'Hôpital's rule bears his name, as for more than two hundred years the *Analyse* contained the first known occurrence of such a rule. [3]

The *Analyse* provided systematic explanations of concepts that Leibniz had presented only in fragmented and often inscrutable papers during the mid 1680s [7]. By demystifying these works, the textbook was vital for the dissemination of calculus across Europe. This is eloquently captured in Fontenelle's 1708 eulogy for L'Hôpital, which describes the early state of the field as a "Cabalistic Science":

> the Geometry of the Infinitely small was still nothing but a kind of Mystery, and, so to speak, a Cabalistic Science shared among five or six people. They often gave their Solutions in the Journals without revealing the Method that produced them, and even when one could discover it, it was only a few feeble rays of this Science that had escaped, and the clouds immediately closed again.
>
> — Fontenelle, 1708 [8]

2.4. L'Hôpital's mistaken solution to Bernoulli's $\frac{0}{0}$ problem

In late 1691 a twenty-four-year-old Johann Bernoulli arrived in Paris [8]. Recognizing the young Swiss mathematician's talent, L'Hôpital engaged him for private lessons in differential and integral calculus. After six months of intensive study in the capital, the Marquis invited Bernoulli to his family estate, the Château d'Oucques, where they spent another three to four months in secluded study during the summer of 1692.

Bernoulli returned to Basel in 1692, and kept up his correspondence with L'Hôpital and other French mathematicians. It is likely that by this point Bernoulli had discovered L'Hôpital's rule, as he challenges the mathematician Pierre Varignon to a problem that can only be solved using it, viz:

Evaluate the following expression when $x = a$:

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

Plugging in numbers directly only leads to an indeterminate form $\frac{0}{0}$:

$$
\begin{aligned}
y(x) &= \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} \\
y(a) &= \frac{\sqrt{2a^3a - a^4} - a\sqrt[3]{a^2a}}{a - \sqrt[4]{aa^3}} \\
&= \frac{\sqrt{2a^4 - a^4} - a\sqrt[3]{a^3}}{a - \sqrt[4]{a^4}} \\
&= \frac{\sqrt{a^4} - aa}{a - a} \\
&= \frac{aa - aa}{a - a} \\
&= \frac{0}{0}
\end{aligned}
$$

L'Hôpital hears of this challenge, and proposes a solution in June 1693 [6, Letter 11]. He uses the difference of two squares to cancel the two instances of the term $(a - a)$:

$$
\begin{aligned}
y &= \frac{aa - aa}{a - a} \\
&= \frac{\cancel{(a - a)}(a + a)}{\cancel{a - a}} = 2a
\end{aligned}
$$

Drawing the curve reveals that L'Hôpital's solution is incorrect:

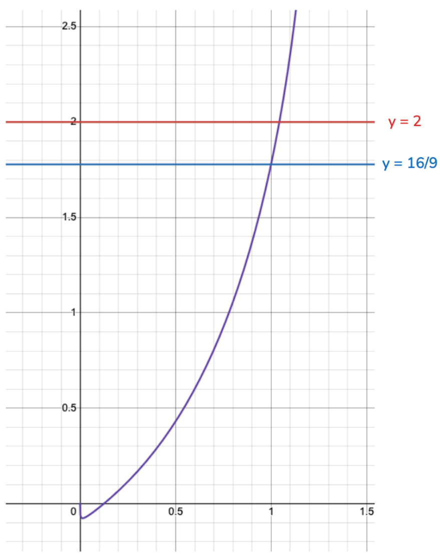

Figure 1: We have plotted L'Hôpital's incorrect answer $y = 2a$ and Bernoulli's correct answer $y = \frac{16}{9}a$ $(a = 1)$.

Indeed, the rules of algebra allow us to rearrange and substitute, but not to rearrange, substitute and then rearrange again, and certainly not to cancel out two terms which both evaluate to 0.

Bernoulli informed L'Hôpital of his error [8], and L'Hôpital pleaded with Bernoulli to tell him the answer in several letters, including L15 and L17 [6]. L15, dated September 2, 1693 ends with the following plea:

> I confess that I did not work very hard to solve the equation
>
> $$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$
>
> where $x = a$. Because I see no hope of success, since all the solutions that first present themselves are not correct, I did not want to waste my time unnecessarily, and I'd prefer to learn it from you if you are willing to share it with me. I finish, Sir, by asking you always to love me and to believe me to be entirely yours
>
> The M. De L'Hôpital

2.5. 'The Contract'

Upon Bernoulli's return to Basel in the fall of 1692, the Marquis forwarded a brilliant solution—privately authored by Bernoulli—to Christiaan Huygens without any mention of his tutor. Huygens, naturally assuming the work was original, showered L'Hôpital with praise. The ensuing dispute between teacher and student served as a wake-up call for the Marquis, who realized that he was not only hopelessly dependent on Bernoulli's genius to maintain his prestige in the Republic of Letters, but that further "intellectual theft" would require a more formal, financial incentive to ensure Bernoulli's continued silence and cooperation. In March 1684, L'Hôpital came with an exceptional proposal: He would pay Bernoulli a retainer of 300 livres (pounds), if Bernoulli would share all his research with L'Hôpital and no one else:

> I will be happy to give you a retainer of 300 pounds, beginning with the first of January of this year . . . I promise shortly to increase this retainer, which I know is very modest, as soon as my affairs are somewhat straightened out . . . I am not so unreasonable as to demand in return all of your time, but I will ask you to give me at intervals some hours of your time to work on what I request and also to communicate to me your discoveries, at the same time asking you not to disclose any of them to others. I ask you even not to send here to Mr. Varignon or to others any copies of the writings you have left with me; if they are published, I will not be at all pleased. Answer me regarding all this [...]
>
> — M. de L'Hôpital,
> Paris, March 17, 1694
> [6, L20]

In case there's any ambiguity about the arrangement, Bernoulli writes in his Letter 28 (emphasis ours):

> [regarding] the discoveries that I have made *on your behalf* and that I will make in the future on the opportunities that you give me, I make you a sacred promise, Sir, to always keep them secret and to let nothing at all out
>
> — Johann Bernoulli
> July 22nd 1694
> [6, L28]

Bernoulli had a famous older brother (Jacob Bernoulli, 1654-1705), but he was young and had not yet made a name for himself in the world of mathematics. Why would he enter into such an agreement in the first place? We surmise that he was induced to this agreement by the money and his own modest financial situation. He had baby on the way, his wife giving birth to his first son the following February. A wealthy Marquis was offering good money for what was essentially a low-effort part-time gig. In those days an unskilled worker in Paris would earn around 250 livres a year[3]

L'Hôpital's dishonesty is apparent by his later promise that he would not publish Bernoulli's work, but keep them secret, stating that he had "No desire to take for himself the honour of these discoveries" (letter 42) [5]. Bernoulli clearly understood the implications of the agreement for his own legacy, as he began a new practice of making copies of his letters to L'Hôpital starting in the summer of 1694 [8].

When the *Analyse* appeared, the arrangement and the payments stopped. Even though it had been published anonymously, it was known that L'Hôpital was the author. Bernoulli realized what he had given away, angrily writing to Varignon on February 26, 1707: "to speak frankly, Mr. de L'Hôpital had no other part in the production of this book than to have translated into French the material that I gave him, for the most part, in Latin" [8]. But because Bernoulli only started crying foul-play after L'Hôpital's death in 1704, nobody believed him [5].

While it would be an exaggeration to label the *Analyse* a mere translation of Bernoulli's private *Lectiones*—as the final textbook is twice as long and enriched with many original practical examples—the core logical framework and geometric illustrations are undeniably lifted from Bernoulli's correspondence. This was confirmed in 1921, when Paul Schafheitlin recovered the Swiss mathematician's original lecture notes in Basel, allowing a full comparison between the works [4]

The next section provides a line-by-line analysis of Bernoulli's Letter to the Marquis, while L'Hôpital's treatment of L'Hôpital's rule is give in Section 3.2.

[3]Around €5250 today [9] and [10], see also [11].

3. Line-by-line commentary

3.1. Bernoulli, 1694 (22 July): Letter 28, Bernoulli to L'Hôpital:

> *Probl.*[72] Given a curve whose nature is expressed by a fraction equal to y, which in a certain case has the numerator and the denominator equal to zero, we wish to find the value, that is to say the magnitude of the ordinate y.

In Bernoulli's statement of the problem, y is unambiguously taken to actually *take on* a particular value when its defining expression has the indeterminate form $\frac{0}{0}$. L'Hôpital's version is more ambiguous on this point, asking what the coordinate y *ought* to be (see Section 3.2).

> *Sol.* Let AEC be the given curve, $AD = x$, $DE = y$, $AB =$ to a constant, such that BC becomes equal to a fraction, the denominator and numerator of which are equal to zero.

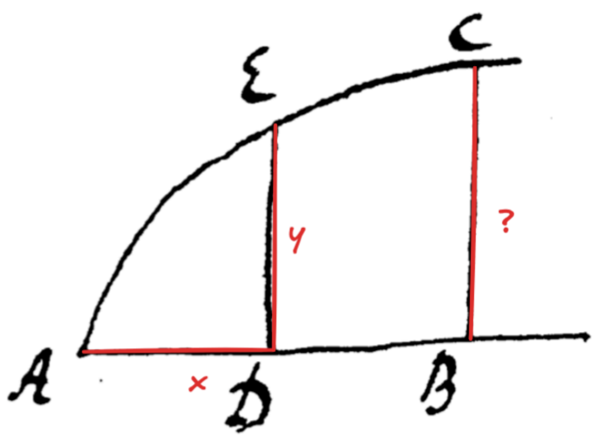

Figure 2: The initial set-up of the problem. BC is the length to be found. It is equal to a fraction, whose denominator and numerator are 0.

Bernoulli continues:

> Therefore, to find the magnitude of the ordinate *BC*, I construct on the same axis *adb* two other curves *aeb* and *αϵb* of such a nature that having taken abscissas equal to *AD* and *ad*, the ordinates *de* are in ratio to the numerator of the general fraction, which expresses the ordinate *DE*, and *dϵ* are in ratio to the denominator of the same fraction.

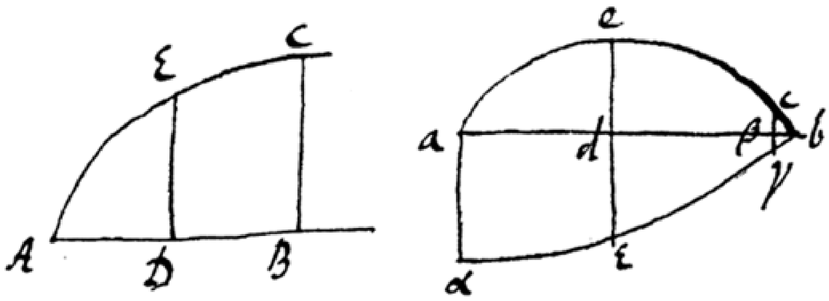

Figure 3: Bernoulli's sketch. On the left hand side we have the function we are considering (AEC), whereas the right hand side shows the numerator and the denominator plotted separately. Seeing as ab is an x-axis it might be tempting to suppose $\alpha\varepsilon b$ to be negative, while aeb is positive. This is unlikely to be what Bernoulli meant, as his original challenge to Varignon (see Section 2.4) has both numerator and denominator positive. Rather, both de and $d\varepsilon$ are positive. We assume that this choice was made to be able to label the curves more clearly.

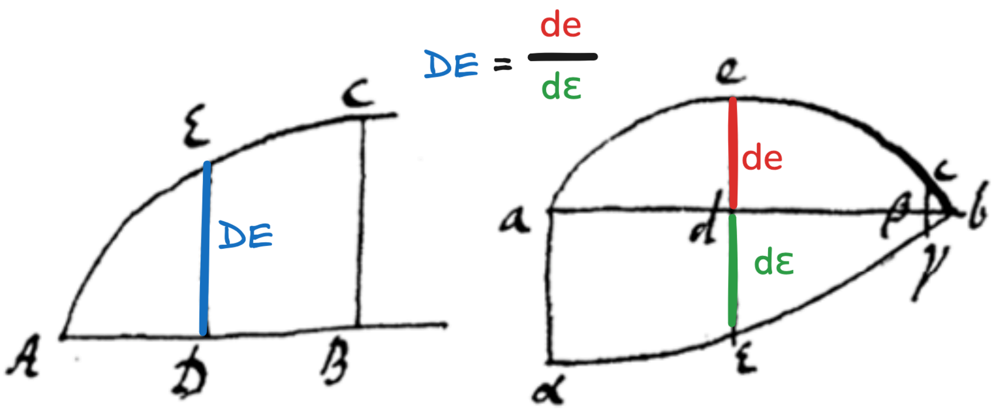

Figure 4: Our annotation of Bernoulli's sketch highlights that the left hand curve is the ratio of the two curves on the right hand side.

We can translate the solution into function notation by proposing three functions, $y(x)$, $f(x)$ and $g(x)$, where $y(x)$ is defined by:

$$y(x) := \frac{f(x)}{g(x)}$$

And:

$$f(a) = g(a) = 0$$

.

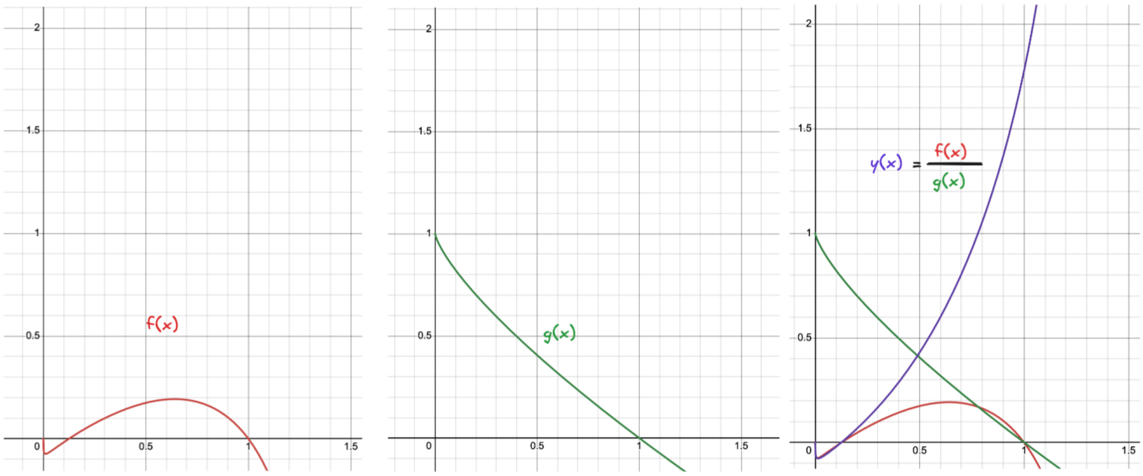

Figure 5: Modern recasting of Bernoulli's set-up for L'Hôpital's problem. For our $f(x)$ and $g(x)$ we have chosen the numerator and denominator of the problem posed to Varignon, see Section 2.4. Curves made using Desmos[4].

However, our modern recasting doesn't take into account that AB is not ab in Figure 3. Instead, it appears that Bernoulli chooses different ordinates x' for his curves $f(x')$ and $g(x')$. Then, $y(x) = \frac{f(x')}{g(x')}$ if $x = \frac{AB}{ab}x'$. This level of generality seems unnecessary, and indeed performing a change of coordinates gives us back $y(x) = \frac{f(x)}{g(x)}$, as Bernoulli states:

> This being done it is clear that *de* divided by $d\epsilon$ may be supposed equal to *DE*.

> The problem therefore reduces to finding the value of *de* divided by $d\epsilon$ in the case that *ab* is equal to *AB*.

In other words, what is $\frac{f(x)}{g(x)}$ when $x = a = 1$?

[4]https://www.desmos.com/calculator/zstzdpaqra

> Now, I see that in this case, *de* and $d\epsilon$ vanish because the two terms of the fraction vanish, and thus the two curves *aeb* and $\alpha\epsilon b$ intersect at the point b.

In other words, $f(a) = 0$ and $g(a) = 0$, and therefore the curves intersect at $x = a = 1$. Next comes the biggest logical leap in the letter:

> *Therefore, we need only take the last differentials*[5] βc *and* $\beta\gamma$*, of which the one divided by the other will tell me the magnitude of BG that I seek*

At this point we expect the reader to be puzzled: Why does the fact that "the two curves aeb and $\alpha\varepsilon b$ intersect at point b?" imply "we need only take the differentials". But we can make sense of this by turning to Postulates 1 and 2 at the start of the *Lectiones*, which as we recall took place in the summer of 1692 at the Marquis' summer palace in Oucques:

> *Postulate 1: Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase*

— *Lectiones de Calculo Differentiales* (1692) [6]

In function notation, Postulate 1 is the (rather strange[6]) statement:

$$f(x) = f(x) + df$$

The first postulate is followed by the second:

> *Postulate 2: Any Curved line consists of infinitely many straight lines, each of which is infinitely small*

— Bernoulli's *Lectiones de Calculo Differentiales*, Postulates[6]

Postulate 2 is illustrated by Figure 6:

[5] *ainsi les deux courbes* aeb *et* $a\varepsilon b$ *se coupent au point b. Il n'y a donc qu'à prendre les dernieres differentielles* βc, $\beta\gamma$, *dont l'une divisée par l'autre me marquera la grandeur cherchée de BC* [12]

[6] It immediately leads to the tricky question: is $df > 0$? The rules of algebra imply $df = 0$, but we seem to need $df > 0$ to be true if we use the infinitesimal to calculate ratios

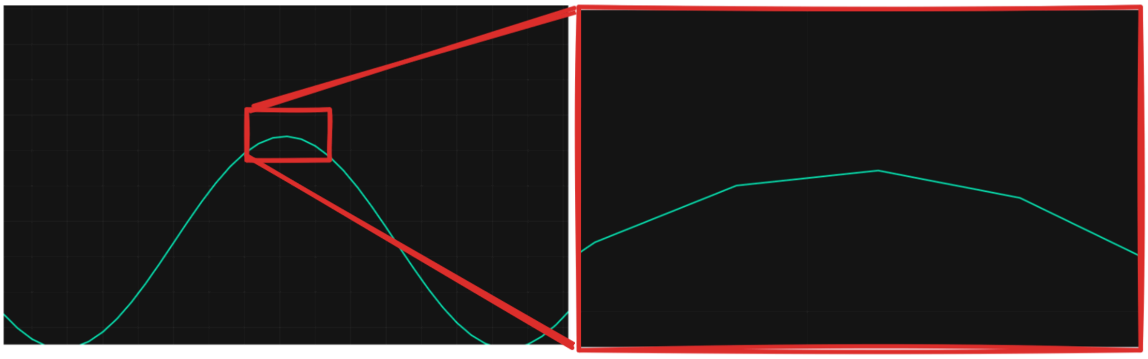

Figure 6: Illustration of Bernoulli's second postulate given in the *Lectiones*.

By Postulate 2, $f(x)$ and $g(x)$ consist of infinitely many straight lines, each of which is infinitely small. By Postulate 1, we can always replace $f(x)$ with $f(x) + df$ or $g(x)$ by $g(x) + dg$ as it suits us. And it seems to suit us particularly well near $x = a$, where we may prefer to consider the behavior of $\frac{f(x)+\delta f}{g(x)+\delta g}$ rather than of $\frac{f(x)}{g(x)}$.

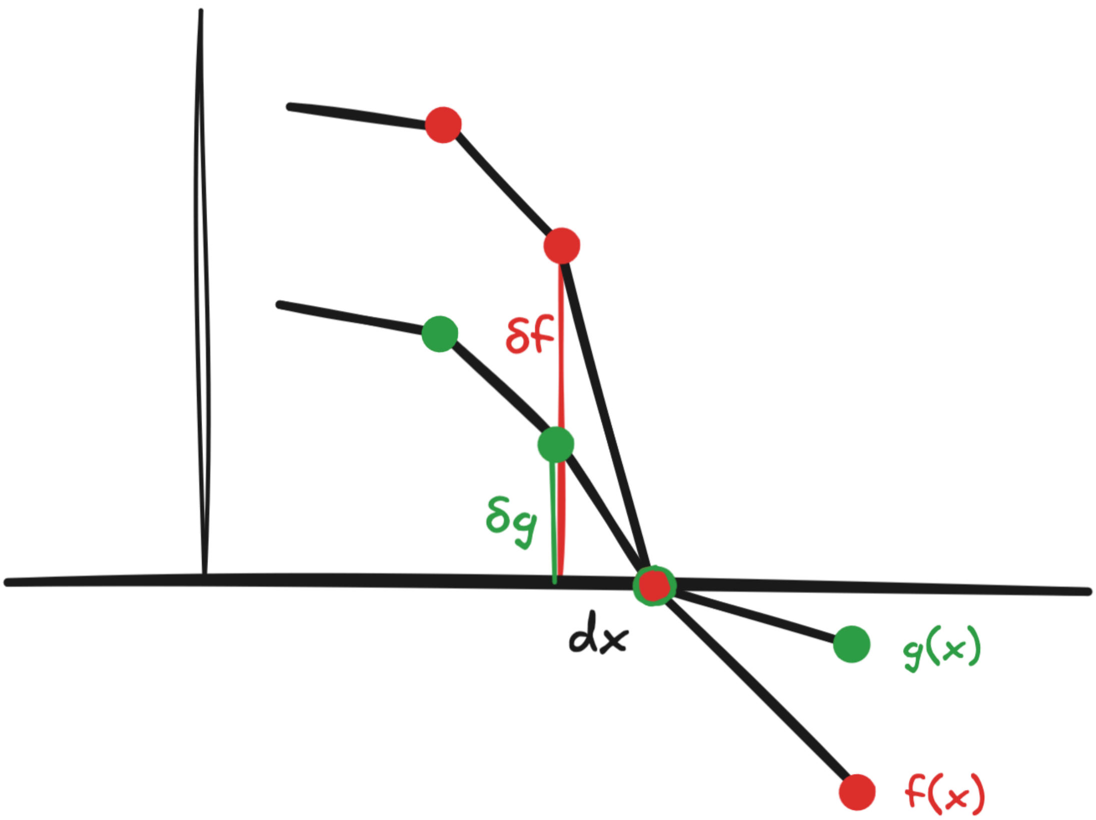

Figure 7: Since curves are really just infinitely many straight lines (Postulate 2), we can zoom in on the intersection point $x = a$, revealing the infinitely small straight lines which make up $g(x)$ and $f(x)$. By Postulate 1, the function $f(x)$ does not increase or decrease if we add δf to it. The same thing goes for $g(x)$.

Presenting the reasoning with terse algebra:

$$y(x)\mid_{x=a} = \frac{f(x)}{g(x)}\mid_{x=a} \overset{\text{Postulate 1}}{=} \frac{f(x)+\delta f}{g(x)+\delta g}\mid_{x=a} = \frac{0+\delta f}{0+\delta g}\mid_{x=a} = \frac{\delta f}{\delta g}\mid_{x=a}$$

Note that at this point Bernoulli speaks of the quotient of differentials $\frac{\delta f}{\delta g}$, not of the derivatives $\frac{f'}{g'}$ as in the standard formulation in today (for a modern statement of L'Hôpital's Rule, see [2, §4.4]).

We hope that this detour into the Postulates of the *Lectiones* has clarified Bernoulli's reasoning. Let us return to the letter at hand, where we now come to the first full statement of what is now known as L'Hôpital's rule:

> *This is what gives me the following general rule:* To find the value of the ordinate of the given curve in the given case we must divide the differential of the numerator of the general fraction by the differential of the denominator; the quotient, after having made x equal to the supposed AB, will be the magnitude of BC *(Figure 3)*

Bernoulli proceeds with two examples. While it might have been effective pedagogy to allow L'Hôpital to attempt the problem—considering he had pressured Bernoulli for a solution for over a year—Bernoulli instead provides the answer himself in just a few lines.

> *Example.* The curve *ACE* has for its equation
>
> $$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} = y.$$

> Thus, if AB is $= a$, we have $BC = \frac{0\,a}{0}$, now we wish to know the true value.

Bernoulli factors out the a in his numerator, obtaining the weird looking $\frac{0a}{0}$ instead of $\frac{0}{0}$. He likely does this due to considerations of dimensionality[7]. See Figure 3 for a reminder of the meaning of AB and BC.

[7]The numerator has dimensionality $[a^2]$ whereas the denominator has dimensionality $[a]$

> According to the rule, I take the differential of the numerator $\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}$, which is
>
> $$= \frac{a^3\,dx - 2x^3\,dx}{\sqrt{2a^3x - x^4}} - \frac{a^2\,dx}{3\sqrt[3]{axx}},$$

> and the differential of the denominator
>
> $$a - \sqrt[4]{ax^3}, \quad \text{which is} \quad \frac{-3a\,dx}{4\sqrt[4]{a^3x}},$$

To a modern observer, Bernoulli appears to apply the chain rule to the numerator and denominator. However, this interpretation is somewhat anachronistic; the formal 'chain rule' belongs to a later, functional era of calculus. Bernoulli instead relied on the rules of differentiation as established in the *Lectiones*.

> having now substituted in the place of x the supposed value a, we find $-\frac{4}{3}a\,dx$ for the first differential and $-\frac{3}{4}\,dx$ for the second one. Therefore,
>
> $$\frac{-\frac{4}{3}a\,dx}{-\frac{3}{4}\,dx} \quad \text{or} \quad \frac{16a}{9} = BC.$$

We have plotted the correct solution (along with L'Hôpital's incorrect one) in Figure 1.

Bernoulli then does another example, which can be solved without the newly-found rule. This allows him to confirm that his result is consistent with existing methods:

> To verify this method, we may take a very easy example such as this one
>
> $$\frac{a\sqrt{ax} - xx}{a - \sqrt{ax}} = y,$$
>
> which we may also solve, although with much difficulty, with common geometry by removing the irrationality; for we will find by either method $BC = 3a$.

We can apply the L'Hôpital - Bernoulli theorem:

$$\begin{aligned} y &= \frac{a\sqrt{ax} - xx}{a - \sqrt{ax}} && \text{differentiate num. and denom.} \\ &= \frac{a\sqrt{a}\frac{1}{2}x^{-\frac{1}{2}} - 2x}{-\sqrt{a}\frac{1}{2}x^{-\frac{1}{2}}} && \text{sub } x = a \\ &= \frac{\frac{1}{2}a - 2a}{-\frac{1}{2}} \\ &= \underline{3a} \end{aligned}$$

This is a lot easier than solving for $\sqrt{ax}$, squaring the answer, substituting $x = a$, and then solving for y which Bernoulli calls "common geometry" but we would now call "laborious algebra"

3.2. L'Hôpital, 1696: Chapter 9 of *Analyse des infinitements petits, pour l'intelligence des lignes courbes*

Two years later, L'Hôpital published the following proof in the final chapter of the *Analyse des infinitement petits, pour l'intelligence des lignes courbes*[8]. This rule will come to be known as L'Hôpital's rule.

> *Chapter 9: The solution of Several Problems that depend upon the Previous Methods*

> **Proposition I.**
>
> **Problem.** [145] (§163) *Let AMD* (see Fig. 9.1) *be a curved line (AP = x, PM = y, and AB = a) such that the value of the ordinate y is expressed by a fraction, in which the numerator and the denominator each becomes zero when x = a, that is to say, when the point P falls on the given point B. We ask what the value of the ordinate BD ought to be.*[1]

To paraphrase with fewer letters, we draw the graph of $y(x) = \frac{f(x)}{g(x)}$, where the numerator $f(x)$ and denominator $g(x)$ are also plotted and they both go to 0 when $x = a$, so $f(a) = g(a) = 0$. $y(a)$ is an indeterminate form of type $\frac{0}{0}$. Nevertheless, examining the graph shows that there is only one reasonable coordinate for y when $x = a$, so the question is, what *ought* to be this y-coordinate? As mentioned in Section 3.1, the word *ought* leaves more ambiguity about the status of the point $\frac{f(a)}{g(a)}$ than Bernoulli's version. A more thorough linguistic analysis on this point would be necessary to find whether the word *ought* may imply a fundamental difference about the existence of the point $\frac{f(a)}{g(a)}$, as opposed to simply referring the quantity that is to be found.

> Let it be understood that there are two curved lines *ANB* and *COB* that have the line *AB* as a common axis, and which are such that the ordinate *PN* expresses the numerator, and the ordinate *PO* the denominator of the general fraction that corresponds to all of the ordinates *PM*, so that $PM = \frac{AB \times PN}{PO}$.

[8]translation: The analysis of infinitesimals for understanding curved lines

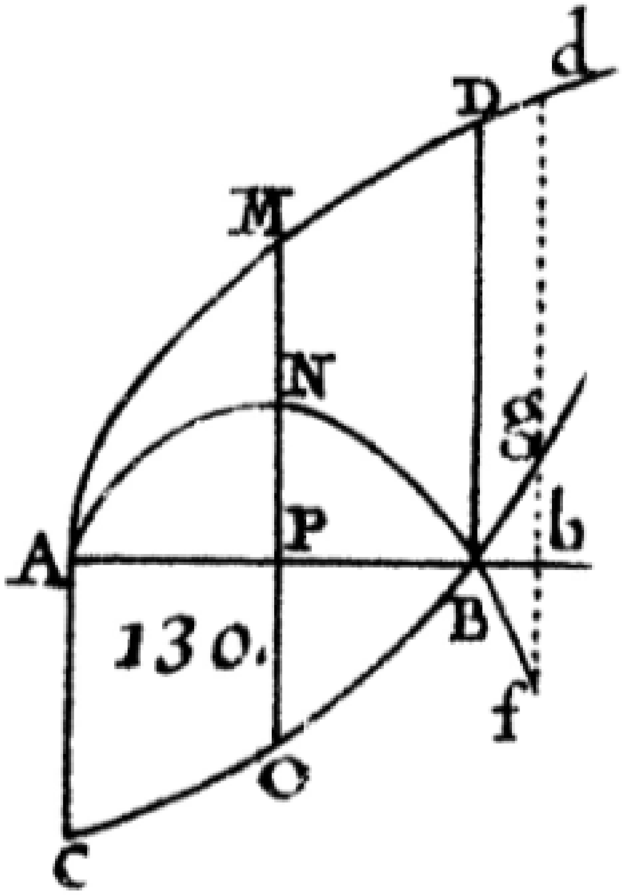

Figure 8: Diagram as it appears in L'Hôpital's *Analyse*. While Bernoulli draws the numerator and denominator separately, L'Hôpital draws them on the same diagram. Compare with Figure 3.

Both OP and PM are intended to be positive values in this illustration, which is why we have drawn $f(x)$ and $g(x)$ as positive as well, in our recasting of the problem:

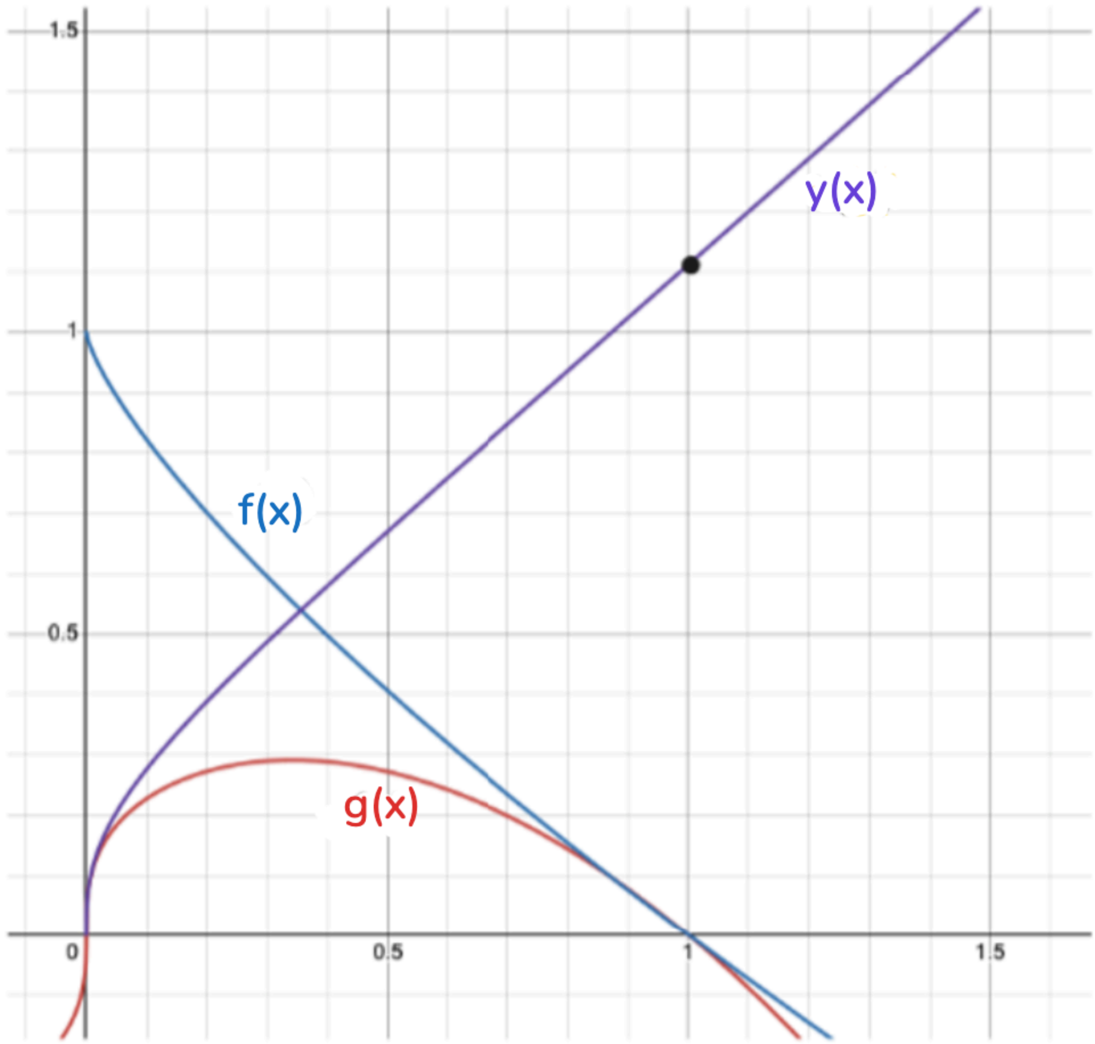

Figure 9: $y(1)$ is undefined but we want to know what it *ought* to be

It seems to us unnecessary to define PM as $\frac{AB \times PN}{PO}$ instead of simply $\frac{PN}{PO}$. Since AB is just a constant (we can call it a), this boils down to identifying PM with our $y(x)$ up to a scaling constant a. Then PN is our numerator $f(x)$ and PO our denominator $g(x)$.

> It is clear that these two curves meet at the point B because, by the assumption, PN and PO each becomes zero when the point P falls on B.

So far the reasoning is exactly like Bernoulli's proof in Section 3.1.

In our construction, $g(x)$ and $f(x)$ meet at $x = a$ because we have defined them (by the assumption) to take on the same value at $x = a$.

> Given this, if we imagine an ordinate bd infinitely close to BD, which meets the curved lines ANB and COB at f and g, then we will have $bd = \frac{AB \times bf}{bg}$, which (see §2) does not differ from BD.

We are now sneaking up on the point $x = a$ from above. We identify the ordinate bd with $y(a + dx)$. The points f and g are simply the values $f(x + dx)$ and $g(x + dx)$. The statement 'bd does not differ from BD', is equivalent to saying '$y(x + dx)$ does not differ from $y(x)$' or $\frac{f(x)+df}{g(x)+dg} = \frac{f(x)}{g(x)}$. To justify this, L'Hôpital explicitly refers us to §2, which is his Postulate I, something Bernoulli neglects to do in his letter (see Section 3.1)

We don't think it is an exaggeration to say that "see §2" is L'Hôpital's only real contribution here. As we saw in Section 3.1, Bernoulli makes a bit of a leap from "the curves intersect at a" to "therefore we should only consider their differentials". Here L'Hôpital fills in the gaps by referring us to his first postulate:

> **Postulate I.**[4] (§2) We suppose that two quantities that differ by an infinitely small quantity may be used interchangeably, or (what amounts to the same [3] thing) that a quantity which is increased or decreased by another quantity that is infinitely smaller than it is, may be considered as remaining the same.
>
> — *L'Hôpital,* Analyse *[4]*

Compare this with Bernoulli's Postulate 1 which we gave on page 11.

L'Hôpital continues:

> It is therefore only a question of finding the ratio of bg to bf.

So the argument goes, just like in Section 3.1: we cannot find $y(a)$ because both $g(a)$ and $f(a)$ are 0, but since $f(x)$ is a curve we can apply Postulate 2 and decompose it into an infinite set of infinitesimal straight line segments. By postulate 2, $f(x)$ does not differ from $f(x) + \delta f$ and $g(x)$ does not differ from $g(x) + \delta g$, so we can simply find $\frac{f(a)+\delta f}{g(a)+\delta g}$ instead of $\frac{f(x)}{g(x)}$.

> Now, it is clear that as the abscissa AP becomes AB, the ordinates PN and PO become null, and that as AP becomes Ab, they become bf and bg.

AB, PN, and PO are shown in Figure 8.

In other words, when $x = a$, $f(x) = g(x) = 0$, and when $x = a + dx$, $f(x) \neq 0$ and $g(x) \neq 0$.

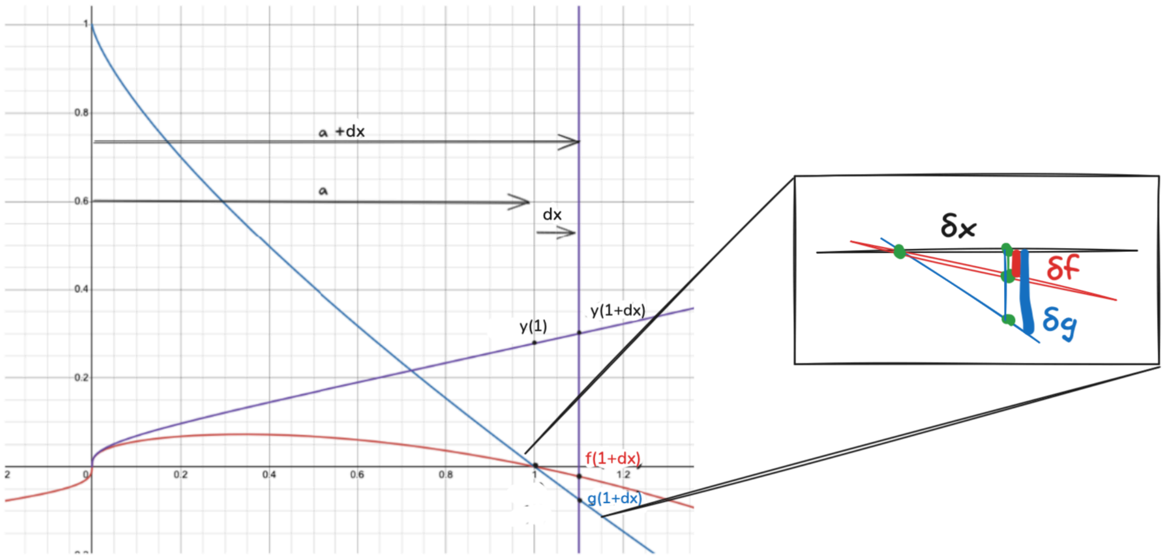

Figure 10: Since $f(x)$ and $g(x)$ intersect at $x = 1$, we can replace their values with the infinitely small quantities δf, δg, by applying Postulates I and II.

> From this, it follows that these ordinates themselves, bf and bg, are the differentials of the ordinates at B and b with respect to the curves ANB and COB.

Now comes L'Hôpital's statement of the rule that has come to bear his name:

> Consequently, if we take the differential of the numerator and we divide it by the differential of the denominator, after [146] having let $x = a = Ab$ or AB, we will have the value that we wish to find for the ordinate bd or BD. This is what we were required to find.

To summarize the proof then:

$$y(a) = \frac{f(a)}{g(a)}$$

But we know that $f(a) = f(a) + \delta f$ by postulate 1, because δf is infinitely small. Same for $g(a)$. Therefore:

$$y(a) = \frac{f(a) + \delta f}{g(a) + \delta g} = \frac{0 + \delta f}{0 + \delta g} = \frac{df}{dg}$$

The following step appears in modern discussions of the rule but is a bit anachronistic:

$$\frac{df}{dg} = \frac{\mathrm{d}f}{\mathrm{d}x} \Big/ \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{f'}{g'}$$

Their calculus was one of *differentials*, not derivatives. And there were no limits.

L'Hôpital provides the correct solution ($y = \frac{16a}{9}$) to the problem he couldn't solve the year before (see Section 2.4). We won't repeat the solution as we looked at it in Section 3.1

Finally, he finishes the section with a slightly different example to Bernoulli's Letter 28.

> *Example II.* (§165) Let[3]
>
> $$y = \frac{aa - ax}{a - \sqrt{ax}}.$$

Finding the differential of the numerator and denominator:

$$\begin{aligned} y(a) &= \frac{-a}{-\sqrt{a}\frac{1}{2}x^{-\frac{1}{2}}} \\ &= 2\frac{a}{\frac{\sqrt{a}}{\sqrt{a}}} \\ &= \underline{2a} \end{aligned}$$

> We find $y = 2a$ when $x = a$.
>
> We might have solved this example without the need of the calculus of differentials in the following way.
>
> Having removed the incommensurables, we will have $aaxx + 2aaxy - axyy - 2a^3x + a^4 + aayy - 2a^3y = 0$, which being divided by $x - a$, reduces to $aax - a^3 + 2aay - ayy = 0$, and substituting a for x, it follows as before that[4] $y = 2a$.

Finally, L'Hôpital explains how we could obtain the same result without his new-found rule through pure algebra (solving for $\sqrt{ax}$, squaring the answer, and then solving for y). Here he diverges (adds to) Bernoulli's exposition as he conscientiousness gives a worked example (whereas Bernoulli just writes that this can be found 'by geometry').

4. Conclusion

We have seen that the earliest proofs of L'Hôpital's rule were the consequences of a differential calculus that relied on two basic postulates. First, that curves are made of infinitely many infinitesimal straight lines, and second, that adding an infinitesimal to a value doesn't change it. This allowed the early founders of calculus to express L'Hôpital's rule without limits and without derivatives.

We have also seen that Bernoulli, not L'Hôpital was the first to propose the rule, and that L'Hôpital paid Bernoulli for the sole rights to know about his discoveries.

Any differences between L'Hôpital's exposition and the one given by Bernoulli are ones of *form*, not *logic*. L'Hôpital exposition has more detail appropriate for a textbook aimed at students seeing calculus for the first time, whereas Bernoulli's contains some short cuts more appropriate for a letter to a student who was quite conversant with the basics.

Bibliography

[1] V. Elgersma and V. Belde, "History of L'Hôpital's Bernoulli's Rule (1692–1696)." [Online]. Available: https://vjbe.net/pdfs/2026/03/31-lhopital-pres.pdf

[2] J. Stewart, *Calculus: Early Transcendentals*, 6th ed. Belmont, CA: Brooks/Cole, Cengage Learning, 2007.

[3] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, "The Solution of Several Problems That Depend upon the Previous Methods," in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 151–168. doi: 10.1007/978-3-319-17115-9_9.

[4] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*. Cham: Springer International Publishing, 2015. doi: 10.1007/978-3-319-17115-9.

[5] C. Truesdell, "The New Bernoulli Edition," *Isis*, vol. 49, no. 1, pp. 54–62, 1958, Accessed: Mar. 20, 2026. [Online]. Available: http://www.jstor.org/stable/226604

[6] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, "Selected Letters from the Correspondence Between the Marquis de L'Hôpital and Johann Bernoulli," in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 233–293. doi: 10.1007/978-3-319-17115-9_12.

[7] G. W. Leibniz, "Nova Methodus pro Maximis et Minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus," *Acta Eruditorum*, vol. 3, pp. 467–473, Oct. 1684.

[8] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, "Translators' Preface," in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, p. vii–ix. doi: 10.1007/978-3-319-17115-9.

[9] R. Edvinsson, "Historical currency converter." Accessed: Mar. 27, 2026. [Online]. Available: https://www.historicalstatistics.org/Currencyconverter.html

[10] I. Webster, "Netherlands Inflation Calculator." Accessed: Mar. 27, 2026. [Online]. Available: https://www.officialdata.org/netherlands/inflation

[11] Wikipedia contributors, "Economic history of France." Accessed: Mar. 26, 2026. [Online]. Available: https://en.wikipedia.org/wiki/Economic_history_of_France

[12] Johann Bernoulli, *Der Briefwechsel von Johann Bernoulli. Band I.* Basel: Birkhäuser, 1955.