P2 – Group 4

Report & Line-by-Line Commentary on Original Sources

L’Hôpital’s Rule[1]

Victor Elgersma[2] & Vera Belde[3]

01 April 2026

[1] “This report constitutes the written component of P2 – Group 4. For the oral presentation materials, see [1].”

[2] v.j.b.elgersma@students.uu.nl
[3] v.m.p.belde@students.uu.nl

Contents

1. L’Hôpital’s Rule: 2
   1.1. Modern Proof 2
2. Historical context 3
   2.1. Bernoulli’s Challenge to Varignon 3
   2.2. L’Hôpital’s *Analyse* and its secret ghost-authorship. 6
   2.3. ‘The Contract’ 6
3. Bernoulli, 1694 (22 July): Letter 28, Bernoulli to L’Hôpital: 8
4. L’Hôpital, 1696: Chapter 9 of *Analyse des infinitements petits, pour l’intelligence des lignes courbes* 15
5. Conclusion 20
Bibliography 21

1. L’Hôpital’s Rule:

Today L'Hôpital’s rule is understood as the following statement:

For a function y constructed by the division of two differentiable functions f and g that tend to 0 as x goes to a certain constant a, then the function itself tends to the division of the differentials of those functions as x goes to a:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In other words, if the numerator and denominator are both 0 at $x = a$, differentiate the top and bottom and try plugging in the values again. You can do this *ad infinitum* as long as the numerator and denominator are differentiable but won’t yield to directly plugging in $x = a$.

While early authors did not speak of limits [2], they did have a version of L'Hôpital’s rule which we will discuss in this report.

1.1. Modern Proof

A typical modern proof (from Wikipedia) of L'Hôpital’s rule is given below:

Suppose that f and g are continuously differentiable at a real number c, that $f(c) = g(c) = 0$, and that $g'(c) \neq 0$. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f(x)-0}{g(x)-0} = \lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}$$

$$= \lim_{x\to c}\frac{\left(\frac{f(x)-f(c)}{x-c}\right)}{\left(\frac{g(x)-g(c)}{x-c}\right)} = \frac{\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)}{\lim_{x\to c}\left(\frac{g(x)-g(c)}{x-c}\right)} = \frac{f'(c)}{g'(c)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

This follows from the difference quotient definition of the derivative. The last equality follows from the continuity of the derivatives at c. The limit in the conclusion is not indeterminate because $g'(c) \neq 0$.

It involves mainly algebraic manipulation, but the steps aren’t exactly well-motivated. How did the founders of calculus approach this problem, without limits or derivatives[4]?

[4]They worked with differentials like $\sqrt{a - x^2}dx$ but not with derivatives like $f'(x)$

2. Historical context

2.1. Bernoulli's Challenge to Varignon

It all began with a challenge. L'Hôpital had heard that his tutor, Johann Bernoulli, had challenged Pierre Varignon[5] to evaluate the following expression when $x = a$:

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

The problem is subtle because plugging in numbers only leads to an indeterminate form $\frac{0}{0}$:

$$\begin{aligned} y(x) &= \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} \\ y(a) &= \frac{\sqrt{2a^3a - a^4} - a\sqrt[3]{a^2a}}{a - \sqrt[4]{aa^3}} \\ &= \frac{\sqrt{2a^4 - a^4} - a\sqrt[3]{a^3}}{a - \sqrt[4]{a^4}} \\ &= \frac{\sqrt{a^4} - aa}{a - a} \\ &= \frac{aa - aa}{a - a} \\ &= \frac{0}{0} \end{aligned}$$

In his 11th letter to Bernoulli, dated June 27th, 1693, L'Hôpital proposes to use the difference of two squares to cancel the two instances of the term $(a - a)$:

$$\begin{aligned} y &= \frac{aa - aa}{a - a} \\ &= \frac{\cancel{(a - a)}(a + a)}{\cancel{a - a}} = 2a \end{aligned}$$

Drawing the curve reveals that L'Hôpital's solution is incorrect:

[5] https://en.wikipedia.org/wiki/Pierre_Varignon

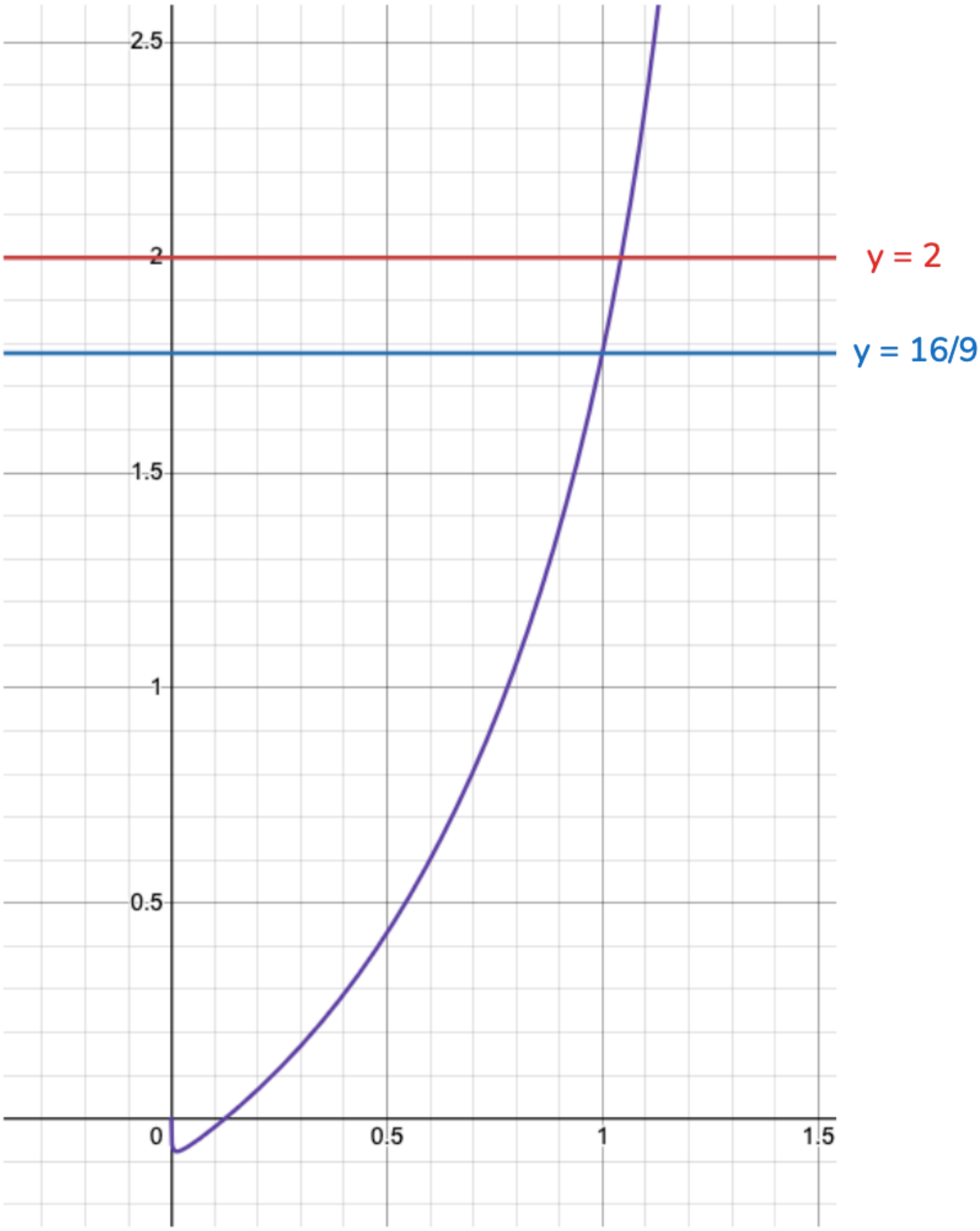

Figure 1: Letting $a = 1$, We have plotted L'Hôpital's incorrect answer $y = 2$ and Bernoulli's correct answer $y = \frac{16}{9}$. For a line by line analysis of Bernoulli's method see Section 3. For L'Hôpital's regurgitation see Section 4

Indeed, the rules of algebra allow us to rearrange and substitute, but not to rearrange, substitute and then rearrange again, and certainly not to cancel out two terms which both evaluate to 0.

Bernoulli informed L'Hôpital of his error [3], and L'Hôpital pleaded with Bernoulli to tell him the answer in several letters, including L15 and L17 [2]. L15, dated September 2, 1693 ends with the following plea:

I confess that I did not work very hard to solve the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

where $x = a$. Because I see no hope of success, since all the solutions that first present themselves are not correct, I did not want to waste my time unnecessarily, and I'd prefer to learn it from you if you are willing to share it with me. I finish, Sir, by asking you always to love me and to believe me to be entirely yours

The M. De L'Hôpital

Bernoulli sends his answer on July 22nd of the following year in Letter 28 (see Section 3 for our line-by-line analysis). This method then reappears almost verbatim in Chapter 9 of L'Hôpital's *Analyse.*

2.2. L’Hôpital’s *Analyse* and its secret ghost-authorship.

Published in 1696 in Paris, the *Analyse des infinitements petits, pour l’intelligence des lignes courbes* is the first known textbook on differential calculus. It was well-known and closely read throughout the 17th century, serving as the first introduction to the subject to many french mathematicians. It is the reason that L’Hôpital’s rule bears the name of the Marquis, as for more than two hundred years it was the first known occurrence of such a rule. [4]

The singular role that the *Analyse* played in spreading the knowledge of calculus is underscored by Fontenelle’s Eulogy for L'Hôpital:

> the Geometry of the Infinitely small was still nothing but a kind of Mystery, and, so to speak, a Cabalistic Science shared among five or six people. They often gave their Solutions in the Journals without revealing the Method that produced them, and even when one could discover it, it was only a few feeble rays of this Science that had escaped, and the clouds immediately closed again.
>
> — Fontenelle, 1708 [3]

How much of the *Analyse* was L'Hôpital’s original work? It was known that L'Hôpital received private lessons from Bernoulli, but until a copy of Johann Bernoulli’s lessons on differential calculus were found in a library in Basel in the 1921 by Paul Schafheitlin, it was impossible to compare the *Analyse* to Bernoulli’s *Lectiones*. Similarly, Bernoulli’s letters to L'Hôpital were unknown until they were rediscovered by Gustav Eneström in 1879. These letters reveal that L'Hôpital paid Bernoulli for priority access to his discoveries, as well as the promise not to publish them or disclose them to anyone else, in what Clifford Truesdell has called one of the most ‘most unusual arrangements in the history of science’ [5].

Not only does Bernoulli’s Letter 28 bear a striking similarity to Chapter 9 of *Analyse*, but it appears that the entirety of the *Analyse* is closely based on the private lectures that Bernoulli gave L'Hôpital.

2.3. ‘The Contract’

L’Hôpital first asked for in-person lessons from Bernoulli, a few times a week. The first six months were spent in Paris, then three or four more months at the chateau of L’Hôpital. But this is not when L’Hôpital’s rule was shared.

After Bernoulli returned to Basel, L’Hôpital sent a solution he learned from Bernoulli to Huygens, without crediting Bernoulli. Huygens assumed the solution was found by L’Hôpital himself. This led to a dispute between L’Hôpital and Bernoulli, but then L’Hôpital came with an exceptional proposal: He would pay Bernoulli a retainer of 300 livres (pounds), if Bernoulli would share all his research with L’Hôpital and no one else:

> I will be happy to give you a retainer of 300 pounds, beginning with the first of January of this year . . . I promise shortly to increase this retainer, which I know is very modest, as soon as my affairs are somewhat straightened out . . . I am not so unreasonable as to demand in return all of your time, but I will ask you to give me at intervals some hours of your time to work on what I request and also to communicate to me your discoveries, at the same time asking you not to disclose any of them to others. I ask you even not to send here to Mr. Varignon or to others any copies of the writings you have left with me; if they are published, I will not be at all pleased. Answer me regarding all this [...]
>
> — L’Hôpital to Bernoulli Letter 20, (Paris, March 17, 1694 [6])

In later letters L'Hôpital promised that he would not publish Bernoulli's work, but keep them secret, stating that he had "No desire to take for himself the honour of these discoveries". After agreeing to the terms of The Contract, Bernoulli began making copies of his letters to L'Hôpital [3]. In case there's any ambiguity about the arrangement, Bernoulli writes in his Letter 28 (emphasis ours):

> [regarding] the discoveries that I have made *on your behalf* and that I will make in the future on the opportunities that you give me, I make you a sacred promise, Sir, to always keep them secret and to let nothing at all out
>
> — Bernoulli to L'Hôpital, Letter 28, July 22nd 1694

Why would Bernoulli enter into such an agreement in the first place? He had a famous older brother[6], but he had not made a name for himself. Clearly he was induced to this agreement by his poor financial position. He had baby on the way - his wife would give birth to his first son in February 1695. And now a thirty-year-old Marquis was offering good money for priority access to his discoveries. In those days an unskilled worker in Paris would earn around 250 livres a year[7], see [9]. Not a bad payment for what is essentially a low-effort part-time gig. Additionally, L'Hôpital had promised to keep his research secret as well, because he had "no desire to take for himself the honor of these discover" (letter 42) [5].

Was it worth it? We do not know as exact sales figures for the *Analyse* were not kept, but it was the standard calculus textbook for most of the 17th century so we imagine it must have made some money. Bernoulli certainly seemed to regret his decision.

When the *Analyse* appeared, the Bernoulli-L'Hôpital arrangement stopped. Even though it had been published anonymously, it was known that L'Hôpital was the author. Bernoulli realized what he had given away, angrily writing to Varignon on February 26, 1707: "to speak frankly, Mr. de L'Hôpital had no other part in the production of this book than to have translated into French the material that I gave him, for the most part, in Latin" [3]. But because Bernoulli only started crying foul-play after L'Hôpital's death in 1704, nobody believed him [5].

It is an exaggeration to say that the *Analyse* is a direct translation of Bernoulli's *Lectiones.* The work is twice as long and contains many examples absent in Bernoulli's notes. But both the structure of the *Analyse* and its core arguments are lifted from the *Lectiones* and Bernoulli's letters. We will see an example of this by comparing Chapter 9 of the *Analyse* (Section 4) with Letter 28 of the Bernoulli-L'Hôpital correspondence (Section 3). We conclude this section by noting that the *Analyse* should be viewed as a co-authorship between Bernoulli and L'Hôpital, in which Bernoulli should be seen as the primary author, and L'Hôpital should be praised mainly for his expository prowess[8] rather than original mathematical thought.

[6]Jacob 1654-1705
[7]Around €5250 today [7] and [8])
[8]This judgment is expressed by Bradley et al in [3]

3. Bernoulli, 1694 (22 July): Letter 28, Bernoulli to L'Hôpital:

> *Probl.*[72] Given a curve whose nature is expressed by a fraction equal to y, which in a certain case has the numerator and the denominator equal to zero, we wish to find the value, that is to say the magnitude of the ordinate y.

In Bernoulli's statement of the problem, y is unambiguously taken to actually *take on* a particular value when its defining expression has the indeterminate form $\frac{0}{0}$. L'Hôpital's version is more ambiguous on this point, asking what the coordinate y *ought* to be, rather than what it actually *is* (see Section 4).

> *Sol.* Let AEC be the given curve, $AD = x$, $DE = y$, $AB =$ to a constant, such that BC becomes equal to a fraction, the denominator and numerator of which are equal to zero.

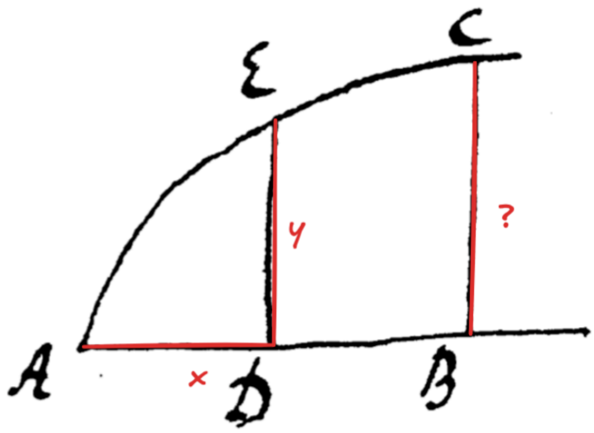

Figure 2: The initial set-up of the problem. BC is the length to be found. It is equal to a fraction, whose denominator and numerator are 0.

Bernoulli continues:

> Therefore, to find the magnitude of the ordinate BC, I construct on the same axis adb two other curves aeb and $\alpha\epsilon b$ of such a nature that having taken abscissas equal to AD and ad, the ordinates de are in ratio to the numerator of the general fraction, which expresses the ordinate DE, and $d\epsilon$ are in ratio to the denominator of the same fraction.

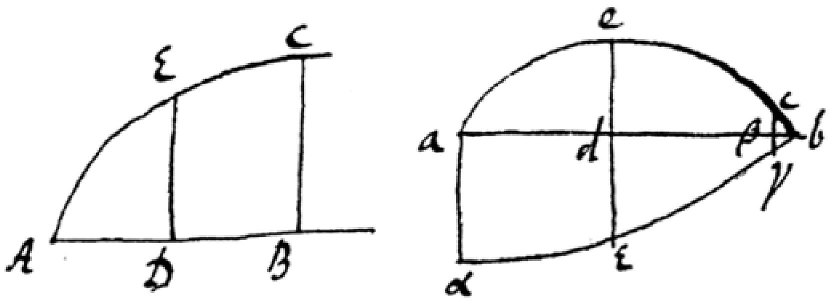

Figure 3: Bernoulli's sketch. On the left hand side we have the function we are considering (AEC), which is composed of ratio of de and $d\varepsilon$ visible on the right hand side. Seeing as ab is an x-axis it might be tempting to suppose $\alpha\varepsilon b$ to be negative, while aeb is positive. This is unlikely to be what Bernoulli meant, as his original challenge to Varignon (see Section 2.1) has both numerator and denominator positive. Rather, both de and $d\varepsilon$ are positive. We assume that this choice was made to be able to label the curves more clearly.

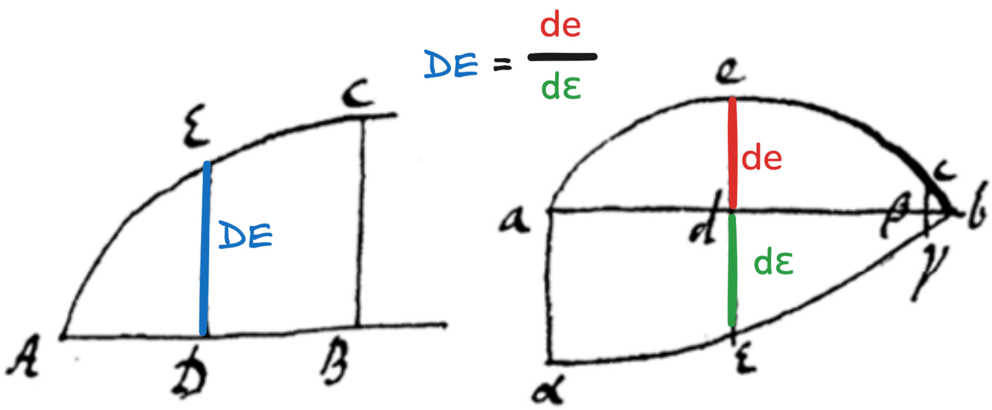

Figure 4: Above, we have annotated Bernoulli's sketch to make it clear that the left hand curve is the ratio of the two curves on the right hand side.

We can translate the solution into function notation by proposing three functions, $y(x)$, $f(x)$ and $g(x)$, where $y(x)$ is defined by:

$$y = \frac{f(x)}{g(x)}$$

And:

$$f(a) = 0 = g(a)$$

.

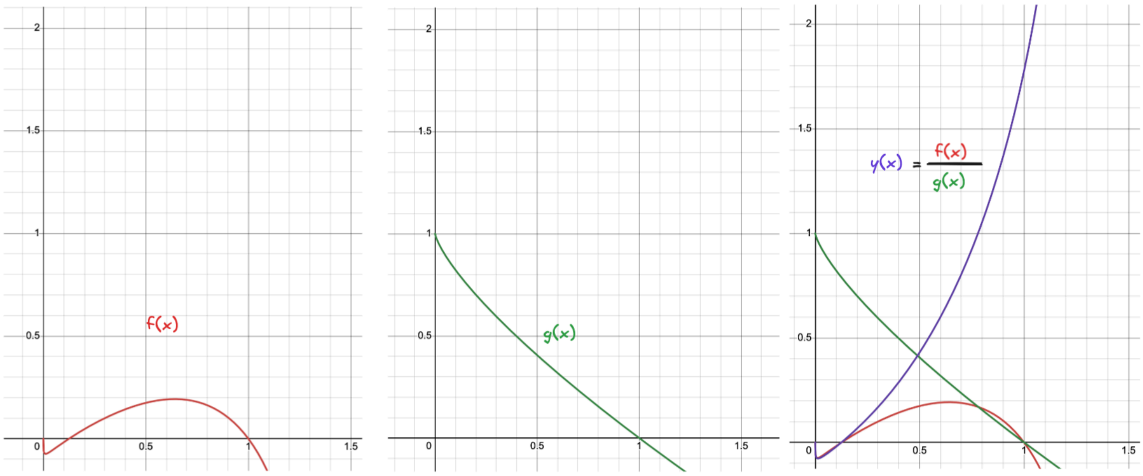

Figure 5: Modern recasting[9] of Bernoulli's set-up for L'Hôpital's problem. For our $f(x)$ and $g(x)$ we have chosen the numerator and denominator of the problem posed to Varignon, see Section 2.1.

Referring to the right-hand side of Figure 3, we find that Bernoulli doesn't draw $f(x)$ and $g(x)$ directly, but prefers to draw the curves $\lambda f(x)$ and $\lambda g(x)$ up to arbitrary scaling constant λ, such that "the ordinates de are in ratio to the numerator of the general fraction [...] and $d\varepsilon$ are in ratio to the denominator of the same fraction". This seems like an unnecessary level of generality.

> This being done it is clear that *de* divided by $d\epsilon$ may be supposed equal to *DE*.

Indeed:

$$y(x) = \frac{\cancel{\lambda} f(x)}{\cancel{\lambda} g(x)}$$

> The problem therefore reduces to finding the value of *de* divided by $d\epsilon$ in the case that *ab* is equal to *AB*.

In other words, what is $\frac{f(x)}{g(x)}$ when $x = a = 1$?

> Now, I see that in this case, *de* and $d\epsilon$ vanish because the two terms of the fraction vanish, and thus the two curves *aeb* and $\alpha\epsilon b$ intersect at the point *b*.

In other words, $f(a) = 0$ and $g(a) = 0$, and therefore the curves intersect at $x = a = 1$. Next comes the biggest logical leap in the letter:

[9] https://www.desmos.com/calculator/zstzdpaqra

[10] *ainsi les deux courbes aeb et aεb se coupent au point b. Il n'y a donc qu'à prendre les dernieres differentielles βc, $\beta\gamma$, dont l'une divisée par l'autre me marquera la grandeur cherchée de BC* [10]

> Therefore, we need only take the last differentials[10] βc and $\beta\gamma$, of which the one divided by the other will tell me the magnitude of BG that I seek

Why does "we need only take the differentials" follow from "the two curves aeb and $\alpha\varepsilon b$ intersect at point b?" We believe that we can make sense of this by turning to Postulates 1 and 2 at the start of the *Lectiones*:

> Postulate 1: Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

— Bernoulli's *Lectiones de Calculo Differentiales*, Postulates[2]

Postulate 1 is the (rather strange to modern eyes) statement:

$$f(x) = f(x) + \delta f$$

> Postulate 2: Any Curved line consists of infinitely many straight lines, each of which is infinitely small

— Bernoulli's *Lectiones de Calculo Differentiales*, Postulates[2]

Postulate 2 is illustrated by Figure 6:

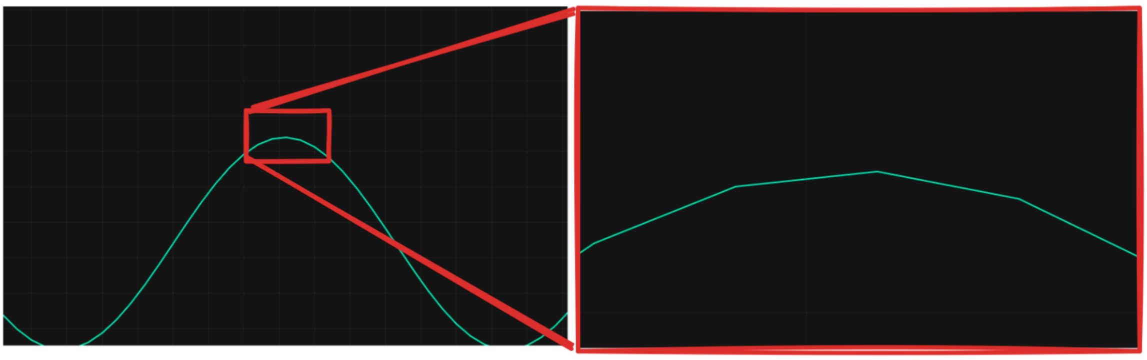

Figure 6: Illustration of Bernoulli's second postulate given in the *Lectiones*.

By Postulate 2, $f(x)$ and $g(x)$ consist of infinitely many straight lines, each of which is infinitely small. By Postulate 1, we can figure out how the fraction $\frac{f(x)}{g(x)}$ behaves near $x = a = 1$ by considering the behavior of $\frac{f(x)+\delta f}{g(x)+\delta g}$.

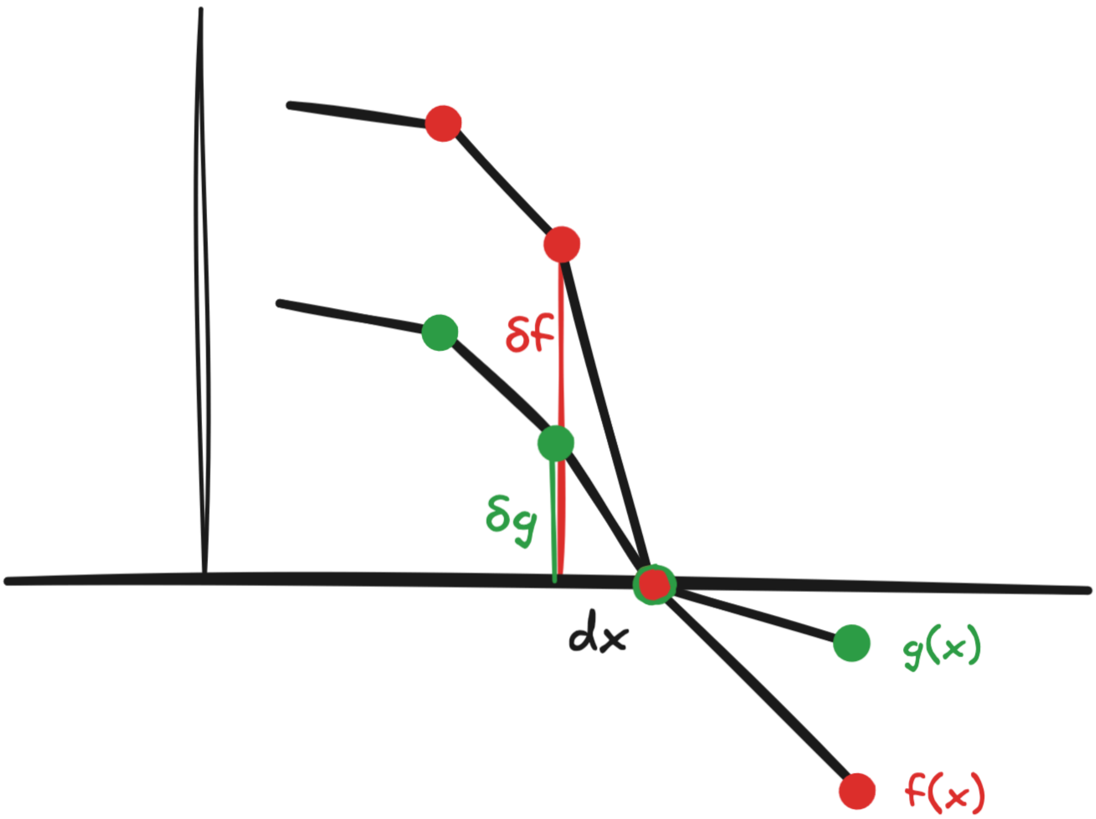

Figure 7: Since curves are really just infinitely many straight lines (Postulate 1), we can zoom in on the intersection point $x = a = 1$, revealing the infinitely small straight lines which make up $g(x)$ and $f(x)$. By Postulate 1, the function $f(x)$ does not increase or decrease if we add δf to it. The same thing goes for $g(x)$.

To summarize his reasoning:

$$y(a) = \frac{f(a)}{g(a)} \overset{\text{Postulate 1}}{=} \frac{f(a) + \delta f}{g(a) + \delta g} = \frac{0 + \delta f}{0 + \delta g} = \frac{\delta f}{\delta g}$$

Note that at this point Bernoulli speaks of the quotient of differentials $\frac{\delta f}{\delta g}$, not of the derivatives $\frac{f'}{g'}$ as in the standard formulation in Section 1.

We hope that this detour into the Postulates of the *Lectiones* has clarified Bernoulli's leap. Let us return to the letter at hand, where we now come to the first full statement of what is now known as L'Hôpital's rule:

> This is what gives me the following general rule: *To find the value of the ordinate of the given curve in the given case we must divide the differential of the numerator of the general fraction by the differential of the denominator; the quotient, after having made x equal to the supposed AB, will be the magnitude of BC* (Figure 3)

Bernoulli then continues to give two examples, the first of which is the original problem that L'Hôpital could not solve himself (see Section 2.1).

Example. The curve ACE has for its equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} = y.$$

Thus, if AB is $= a$, we have $BC = \frac{0\,a}{0}$, now we wish to know the true value.

See Figure 3 for a reminder of the meaning of AB and BC.

Bernoulli factors out the a in his numerator, obtaining the weird looking $\frac{0a}{0}$ instead of $\frac{0}{0}$. He likely does this due to considerations of dimensionality[11]

According to the rule, I take the differential of the numerator $\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}$, which is

$$= \frac{a^3\,dx - 2x^3\,dx}{\sqrt{2a^3x - x^4}} - \frac{a^2\,dx}{3\sqrt[3]{axx}},$$

and the differential of the denominator

$$a - \sqrt[4]{ax^3}, \quad \text{which is} \quad \frac{-3a\,dx}{4\sqrt[4]{a^3x}},$$

Here Bernoulli applies the chain rule for differentiation to the numerator and the rule for differentiating surds to the denominator. He describes these in the earlier chapters of the *Lectiones* [11]. Bernoulli would not have applied this rule to, say, $\sin(x)/x$ because his (Leibnizian) calculus contained no transcendental functions.

having now substituted in the place of x the supposed value a, we find $-\frac{4}{3}a\,dx$ for the first differential and $-\frac{3}{4}\,dx$ for the second one. Therefore,

$$\frac{-\frac{4}{3}a\,dx}{-\frac{3}{4}\,dx} \quad \text{or} \quad \frac{16a}{9} = BC.$$

We have plotted the correct solution (along with L'Hôpital's incorrect one) in Figure 1.

[11]The numerator has dimensionality $[a^2]$ whereas the denominator has dimensionality $[a]$

Bernoulli does another example, which can be solved without the newly-found rule.

> To verify this method, we may take a very easy example such as this one
>
> $$\frac{a\sqrt{ax} - xx}{a - \sqrt{ax}} = y,$$
>
> which we may also solve, although with much difficulty, with common geometry by removing the irrationality; for we will find by either method $BC = 3a$.

We can apply the L'Hôpital - Bernoulli theorem:

$$\begin{aligned} y &= \frac{a\sqrt{ax} - xx}{a - \sqrt{ax}} \\ &= \frac{a\sqrt{a}\frac{1}{2}x^{-\frac{1}{2}} - 2x}{-\sqrt{a}\frac{1}{2}x^{-\frac{1}{2}}} \\ &= \frac{\frac{1}{2}a - 2a}{-\frac{1}{2}} \\ &= \underline{3a} \end{aligned}$$

For Bernoulli, this example could also be solved using "common geometry" by "removing the irrationality"[12].

[12]Perhaps he means collecting the term $\sqrt{ax}$, squaring it, substituting $x = a$ and then solving for y. Indeed this would appear pretty laborious

4. L'Hôpital, 1696: Chapter 9 of *Analyse des infinitements petits, pour l'intelligence des lignes courbes*

Two years later, L'Hôpital published the following proof in the final chapter of the *Analyse des infinitement petits, pour l'intelligence des lignes courbes*[13]. This rule will come to be known as L'Hôpital's rule.

> Chapter 9: The solution of Several Problems that depend upon the Previous Methods

> **Proposition I.**
>
> **Problem.** [145] (§163) *Let AMD* (see Fig. 9.1) *be a curved line ($AP = x$, $PM = y$, and $AB = a$) such that the value of the ordinate y is expressed by a fraction, in which the numerator and the denominator each becomes zero when $x = a$, that is to say, when the point P falls on the given point B. We ask what the value of the ordinate BD ought to be.*[1]

To paraphrase with fewer letters, we draw the graph of $y(x) = \frac{f(x)}{g(x)}$, where the numerator $f(x)$ and denominator $g(x)$ are also plotted and they both go to 0 when $x = a = 1$, so $f(a) = g(a) = 0$. $y(a)$ is an indeterminate form of type $\frac{0}{0}$. Nevertheless, examining the graph shows that there is only one reasonable coordinate for y when $x = a = 1$, so the question is, what *ought* to be this y-coordinate?

> Let it be understood that there are two curved lines *ANB* and *COB* that have the line *AB* as a common axis, and which are such that the ordinate *PN* expresses the numerator, and the ordinate *PO* the denominator of the general fraction that corresponds to all of the ordinates *PM*, so that $PM = \frac{AB \times PN}{PO}$.

[13] translation: The analysis of infinitesimals for understanding curved lines

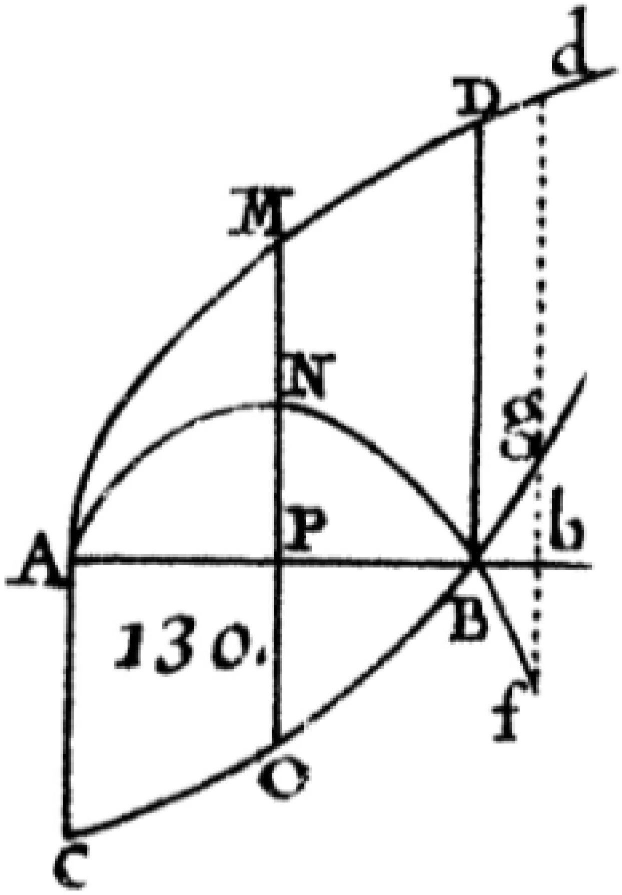

Figure 8: Diagram as it appears in L'Hôpital's *Analyse*. Compare with Figure 3 by Bernoulli

In the drawing by L'Hôpital, what may appear to modern eyes like a curve with negative y-values is not meant in this way. Both OP and PM are intended to be positive values in this illustration, which is why we have drawn $f(x)$ and $g(x)$ as positive as well.

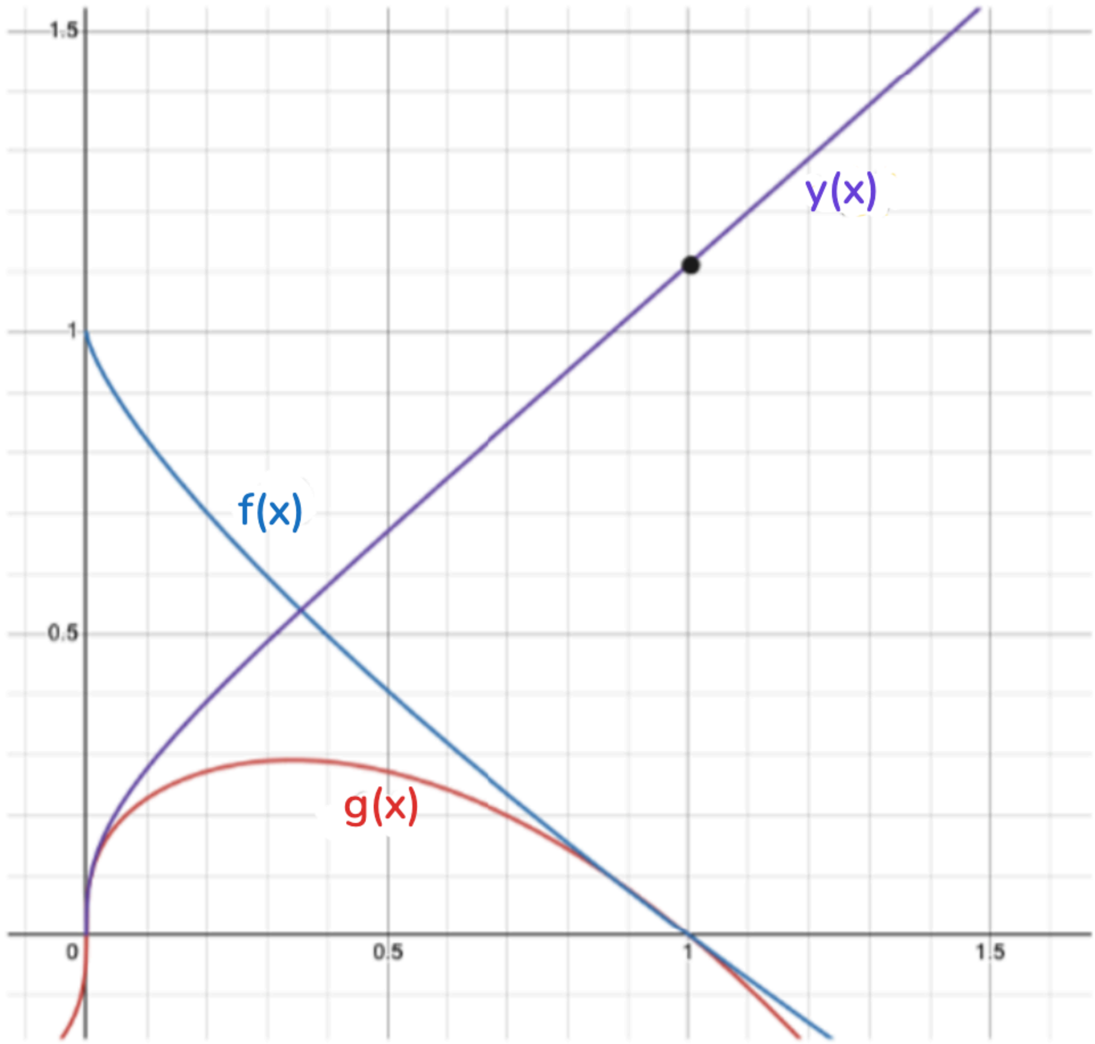

Figure 9: $y(1)$ is undefined but we want to know what it *ought* to be

It seems to us unnecessary to define PM as $\frac{AB \times PN}{PO}$ instead of simply $\frac{PN}{PO}$. Since AB is just a constant (we can call it a), this boils down to identifying PM with our $y(x)$ up to a scaling constant a. Then PN is our numerator $f(x)$ and PO our denominator $g(x)$.

> It is clear that these two curves meet at the point *B* because, by the assumption, *PN* and *PO* each becomes zero when the point *P* falls on *B*.

So far the reasoning is exactly like Bernoulli's proof in Section 3.

In our construction, $g(x)$ and $f(x)$ meet at $x = a$ because we have defined them (by the assumption) to take on the same value at $x = a$.

> Given this, if we imagine an ordinate *bd* infinitely close to *BD*, which meets the curved lines *ANB* and *COB* at *f* and *g*, then we will have $bd = \frac{AB \times bf}{bg}$, which (see §2) does not differ from *BD*.

We are now sneaking up on the point $x = a$ from above. We identify the ordinate bd with $y(a + dx)$. The points f and g are simply the values $f(x + dx)$ and $g(x + dx)$. The statement 'bd does not differ from BD', is equivalent to saying '$y(x + dx)$ does not differ from $y(x)$' or $\frac{f(x+dx)}{g(x+dx)} = \frac{f(x)}{g(x)}$. To justify this, L'Hôpital refers us to §2, which is his Postulate I.

We don't think it is an exaggeration to say that "see §2" is L'Hôpital's only real contribution here. As we saw in Section 3, Bernoulli makes a bit of a leap from "the curves intersect at a" to "therefore we should only consider their differentials". Here L'Hôpital fills in the gaps by referring us to his first postulate:

> **Postulate I.**[4] (§2) We suppose that two quantities that differ by an infinitely small quantity may be used interchangeably, or (what amounts to the same [3] thing) that a quantity which is increased or decreased by another quantity that is infinitely smaller than it is, may be considered as remaining the same.
>
> — L'Hôpital, *Analyse* [6]

Compare this with Bernoulli's Postulate 1 which we give on page 11.

L'Hôpital continues:

> It is therefore only a question of finding the ratio of *bg* to *bf*.

So the argument goes, just like in Section 3: we cannot find $y(a)$ because both $g(a)$ and $f(a)$ are 0, but since $f(x)$ is a continuous function we can apply Postulate 2 and decompose it into an infinite set of infinitesimal straight lines. By postulate 2, $f(x)$ does not differ from $f(x) + \delta f$ and $g(x)$ does not differ from $g(x) + \delta g$, so we can simply find $\frac{f(a)+\delta f}{g(a)+\delta g}$ instead of $\frac{f(x)}{g(x)}$.

> Now, it is clear that as the abscissa *AP* becomes *AB*, the ordinates *PN* and *PO* become null, and that as *AP* becomes *Ab*, they become *bf* and *bg*.

See Figure 8

In other words, when $x = a$, $f(x) = g(x) = 0$, and when $x = a + dx$, $f(x) \neq 0$ and $g(x) \neq 0$. bf and bg are non-zero, as is clear from Figure 8.

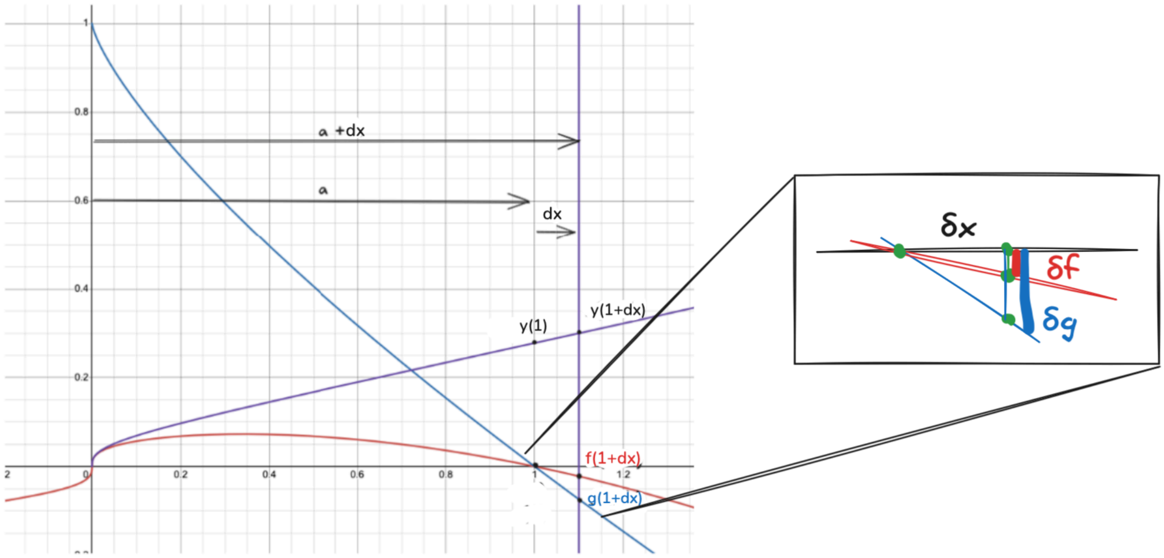

Figure 10: Since $f(x)$ and $g(x)$ intersect at $x = 1$, we can replace their values with the infinitely small quantities δf, δg, by applying Postulates I and II.

> From this, it follows that these ordinates themselves, *bf* and *bg*, are the differentials of the ordinates at *B* and *b* with respect to the curves *ANB* and *COB*.

Now comes L'Hôpital's statement of L'Hôpital's theorem:

> Consequently, if we take the differential of the numerator and we divide it by the differential of the denominator, after [146] having let $x = a = Ab$ or *AB*, we will have the value that we wish to find for the ordinate *bd* or *BD*. This is what we were required to find.

To summarize the proof then:

$$y(a) = \frac{f(a)}{g(a)}$$

But we know that $f(a) = f(a) + \delta f$ by postulate 1, because δf is infinitely small. Same for $g(a)$. Therefore:

$$y(a) = \frac{f(a) + \delta f}{g(a) + \delta g} = \frac{0 + \delta f}{0 + \delta g} = \frac{df}{dg}$$

The next step,

$$\frac{df}{dg} = \frac{\mathrm{d}f}{\mathrm{d}x} \Big/ \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{f'}{g'}$$

is an anachronism, as this notation does not appear. Their calculus was one of *differentials*, not derivatives. And there were no limits. Therefore any similarity with the modern proof in Section 1.1 is superficial.

L'Hôpital provides the correct solution (16/9) to the problem he couldn't solve the year before (see Section 2.1). We won't repeat the solution as we looked at it in Section 3

Finally, he finishes the section with a slightly different example to Bernoulli's Letter 28.

> *Example II.* (§165) Let[3]
>
> $$y = \frac{aa - ax}{a - \sqrt{ax}}.$$

Finding the differential of the numerator and denominator:

$$\begin{aligned} y(a) &= \frac{-a}{-\sqrt{a}\frac{1}{2}x^{-\frac{1}{2}}} \\ &= 2\frac{a}{\frac{\sqrt{a}}{\sqrt{a}}} \\ &= \underline{2a} \end{aligned}$$

> We find $y = 2a$ when $x = a$.
>
> We might have solved this example without the need of the calculus of differentials in the following way.
>
> Having removed the incommensurables, we will have $aaxx + 2aaxy - axyy - 2a^3x + a^4 + aayy - 2a^3y = 0$, which being divided by $x - a$, reduces to $aax - a^3 + 2aay - ayy = 0$, and substituting a for x, it follows as before that[4] $y = 2a$.

Finally, L'Hôpital explains how we could obtain the same result without his new-found rule through pure algebra (solving for $\sqrt{ax}$, squaring the answer, and simplifying). Here he shows his conscientiousness as an expositor, as he is a more careful to explain all the steps in his logic than Bernoulli is (who just writes that this can be found 'by geometry').

5. Conclusion

We have seen that the earliest proofs of L'Hôpital's rule were the consequences of a differential calculus that relied on two basic postulates: 1) Curves are made of infinitely many infinitesimal straight lines, and 2) adding an infinitesimal to a value doesn't change it. This allowed the early founders of calculus to express L'Hôpital's rule without limits and without derivatives.

We have also seen that Bernoulli, not L'Hôpital was the first to propose the rule, and that L'Hôpital paid Bernoulli for the sole rights to know about his discoveries.

Bibliography

[1] V. Elgersma and V. Belde, “History of L'Hôpital’s Bernoulli’s Rule (1692–1696).” [Online]. Available: https://vjbe.net/pdfs/2026/03/31-lhopital-pres.pdf

[2] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, “Selected Letters from the Correspondence Between the Marquis de L'Hôpital and Johann Bernoulli,” in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 233–293. doi: 10.1007/978-3-319-17115-9_12.

[3] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, “Translators' Preface,” in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, p. vii–ix. doi: 10.1007/978-3-319-17115-9.

[4] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, “The Solution of Several Problems That Depend upon the Previous Methods,” in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 151–168. doi: 10.1007/978-3-319-17115-9_9.

[5] C. Truesdell, “The New Bernoulli Edition,” *Isis*, vol. 49, no. 1, pp. 54–62, 1958, Accessed: Mar. 20, 2026. [Online]. Available: http://www.jstor.org/stable/226604

[6] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*. Cham: Springer International Publishing, 2015. doi: 10.1007/978-3-319-17115-9.

[7] R. Edvinsson, “Historical currency converter.” Accessed: Mar. 27, 2026. [Online]. Available: https://www.historicalstatistics.org/Currencyconverter.html

[8] I. Webster, “Netherlands Inflation Calculator.” Accessed: Mar. 27, 2026. [Online]. Available: https://www.officialdata.org/netherlands/inflation

[9] Wikipedia contributors, “Economic history of France.” Accessed: Mar. 26, 2026. [Online]. Available: https://en.wikipedia.org/wiki/Economic_history_of_France

[10] Johann Bernoulli, *Der Briefwechsel von Johann Bernoulli. Band I.* Basel: Birkhäuser, 1955.

[11] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, “Bernoulli's Lectiones de Calculo Differentialis,” in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 187–231. doi: 10.1007/978-3-319-17115-9_11.