

History of L'Hôpital's Bernoulli's Rule (1692-1696)

History of Mathematics Seminar

P2 Group 4

March 31, 2026

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Contents

Overview

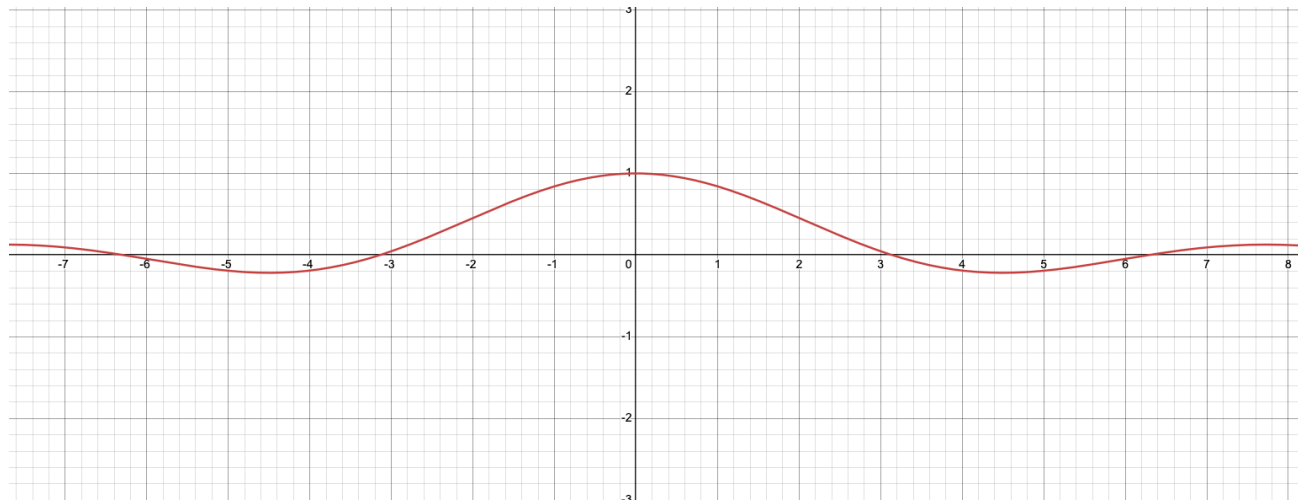
- Reminder: L'Hôpital's rule in 2026
- 1692: The Marquis de L'Hôpital's receives private tuition from Johann Bernoulli
 - Postulates I and II of Bernoulli's *Lectiones de Calculo Differentialibus* (1692)
- 1693:
 - Bernoulli's challenges Varignon to evaluate an expression of indeterminate form $\frac{0}{0}$
 - L'Hôpital proposes an incorrect solution
- 1694:
 - L'Hôpital pays Bernoulli for exclusive access his research (March 1694)
 - Bernoulli shares L'Hôpital's rule with L'Hôpital (July 1694)
- 1696:
 - L'Hôpital publishes *Analyse des infiniment petits pour l'intelligence des lignes courbes* (1696)¹
 - L'Hôpital's rule appears in Chapter 9.
- Conclusion

¹translation: The analysis of infinitesimals for understanding curved lines

L'Hôpital's rule in 2026

Example: What is $\frac{\sin(x)}{x}$ at $x = 0$?

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Direct Substitution

Try plugging in $x = 0$:

$$\frac{\sin(0)}{0} = \frac{0}{0}$$



Indeterminate form!

Applying L'Hôpital's Rule:

Differentiate numerator and denominator:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(x)}{x} &\stackrel{\text{L'Hôpital's Rule}}{=} \lim_{x \rightarrow 0} \frac{\sin(x)'}{x'} = \lim_{x \rightarrow 0} \frac{\cos(x)}{1} \\ &= \frac{\cos(0)}{1} \\ &= \underline{1}\end{aligned}$$

Applying L'Hôpital's Rule:

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Limit exists and is finite $\rightarrow \frac{\sin(x)}{x}$ is said to have a 'removable discontinuity' at $x = 0$

L'Hôpital's Rule: Stated Formally in 2026

Suppose² f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a):

Suppose that:

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

Or

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

Then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Assuming the limit on the right exists (or is $\pm\infty$)

²Stewart, *Calculus: Early Transcendentals*.

	Modern Calculus (2026)	Leibnizian Calculus (1692)
...is about:	Functions and their derivatives	Variable quantities and their differentials
Differentials are...	Part of notation, eg $\int dx$ or $\frac{d}{dx}$	Infinitely small quantities; $x \rightarrow x + dx$ for the purposes of calculation
Derivatives are:	Defined via limits	(simply) ratios of differentials
Functions are	Formally defined as maps $f : A \rightarrow B$	not clearly defined
Limits	Fundamental	Not clearly defined
$\sin(x)$ is	Standard Function	Not a standard function; Geometrically understood



Johann Bernoulli (1667-1748)



Le Marquis de L'Hôpital (1661 – 1704)

**1692: Johann Bernoulli tutors the
Marquis de L'Hôpital**

Postulates I and II of the *Lectiones* (1692)

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- *Lectiones de Calculo Differentialiales* (1692)

Postulate I

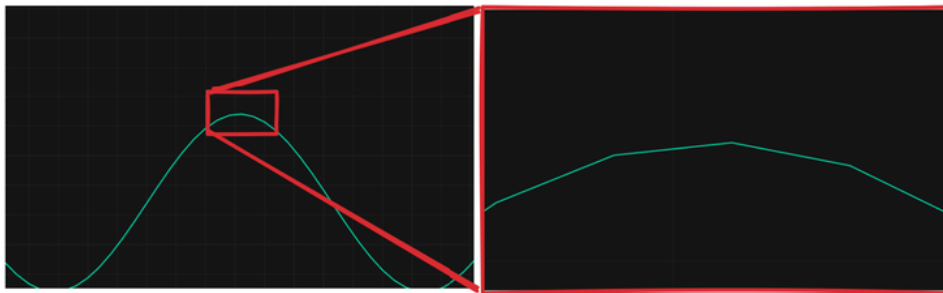
Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase³

✓ $x \rightarrow dx + x$

³Bradley et al., “Bernoulli’s *Lectiones De Calculo Differentialis*”.

Postulate II

Any Curved line consists of infinitely many straight lines, each of which is infinitely small⁴



⁴Bradley et al., “Bernoulli’s Lectiones De Calculo Differentialis”.

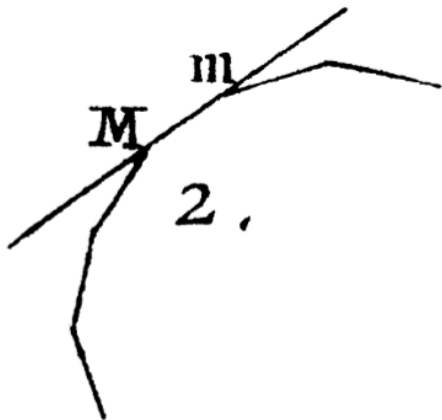
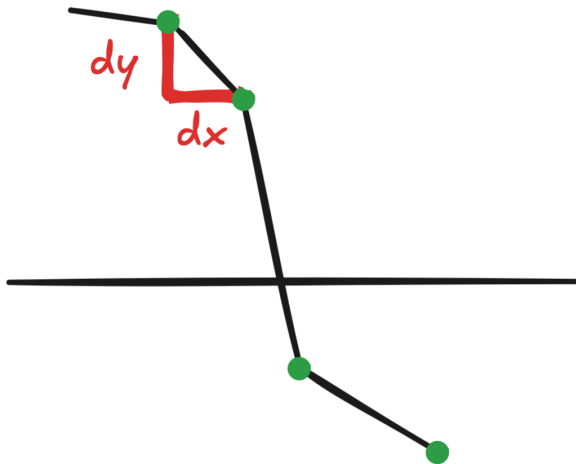


Fig. 2 Postulate II

Postulate II (iii)

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$$y = a - \sqrt{ax^3}$$
$$dy = -\frac{3}{2}\sqrt{ax} \, dx$$

If curves are made of infinitely many straight line segments, then differentials are simply the sides of the triangles making up these segments.

1693

L'Hôpital hears of his teacher's challenge to Varignon

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L'Hôpital to Bernoulli (June 27th, 1693):

I saw

Mr. Varignon who seems to me to be a very good friend of yours, and he gave me a small piece of paper on which had the following question, which he told me you had sent to him. Let the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

express the nature of a curve whose abscissa is x and whose ordinate is y . We wish to know the value of y when x becomes equal to the constant a . Solution. $y = 2a$, because in this case²²

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \frac{aa - aa}{a - a} = a + a.$$

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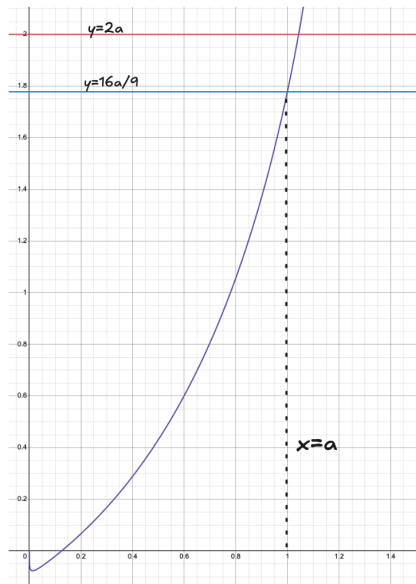
express the nature of a curve whose abscissa is x and whose ordinate is y . We wish to know the value of y when x becomes equal to the constant a . Solution. $y = 2a$, because in this case²²

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \frac{aa - aa}{a - a} = a + a.$$

difference of two squares

$$\begin{aligned} \frac{aa - aa}{a - a} &= \frac{(a \cancel{- a})(a + a)}{a \cancel{- a}} \\ &= \underline{2a} \end{aligned}$$

Answer is incorrect



$y = 2a$ is incorrect

L'Hôpital to Bernoulli: 🙏🙏🙏

Letter 15 (September 2, 1693)

I confess that I did not work very hard to solve the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

where $x = a$. Because I see no hope of success, since all the solutions that first present themselves are not correct, I did not want to waste my time unnecessarily, and I'd prefer to learn it from you if you are willing to share it with me. I finish, Sir, by asking you always to love me and to believe me to be entirely yours

The M. De L'Hôpital

Mrs. de L'Hôpital is your servant.

L'Hôpital to Bernoulli: 🙏🙏🙏 (ii)

Letter 17: (December 2, 1693):

...But after that, I beg you to remember your promise on the tangent to the tractrices and the solution to the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y \quad \text{where } x = a$$

where I am sure I will see something quite unique and very beautiful, as in all your inventions.

1694

L'Hôpital: 💰 🤔 😈 ?

I will happily give you a pension of three hundred *livres*, which will begin the first of January of this present year [...] I will ask you at intervals to give me a few hours of your time, to work on what I will ask you and also to communicate your discoveries to me, while asking you at the same time not to share any of them with others

— L'Hôpital to Bernoulli Letter 20: March 17, 1694⁵

⁵Bradley et al., “Selected Letters from the Correspondence between the Marquis De L'hôpital and Johann Bernoulli”.

[regarding] the discoveries that I have made *on your behalf* and that I will make in the future on the opportunities that you give me, I make you a sacred promise, Sir, to always keep them secret and to let nothing at all out

— Bernoulli to L'Hôpital Letter 28, July 22nd 1694⁶

⁶Bradley et al., “Selected Letters from the Correspondence between the Marquis De L'hôpital and Johann Bernoulli”.

Bernoulli explains L'Hôpital's rule to L'Hôpital

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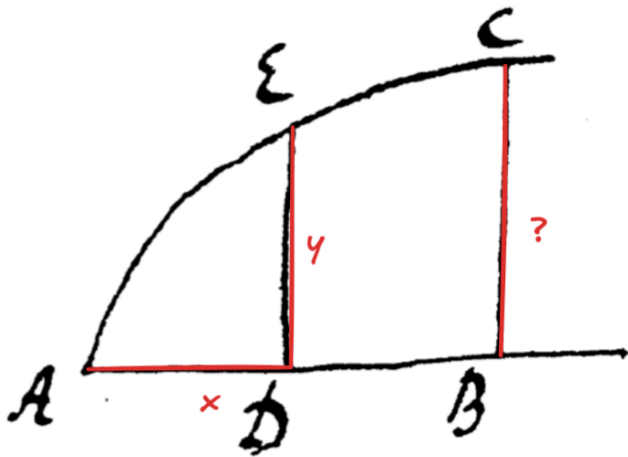
Letter 28: 22 July, 1694:⁷

*Probl.*⁷² Given a curve whose nature is expressed by a fraction equal to y , which in a certain case has the numerator and the denominator equal to zero, we wish to find the value, that is to say the magnitude of the ordinate y .

⁷Bradley et al., "Selected Letters from the Correspondence between the Marquis De L'hôpital and Johann Bernoulli".

Bernoulli explains L'Hôpital's rule to L'Hôpital (ii)

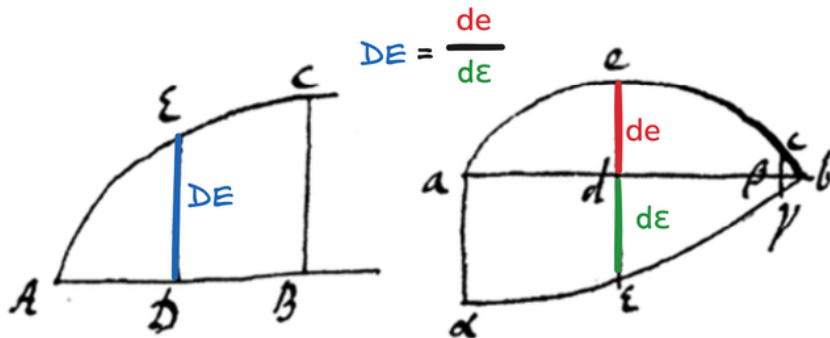
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Sol. Let AEC be the given curve, $AD = x$, $DE = y$, $AB =$ a constant such that BC becomes equal to a fraction, the denominator and numerator which are equal to zero.

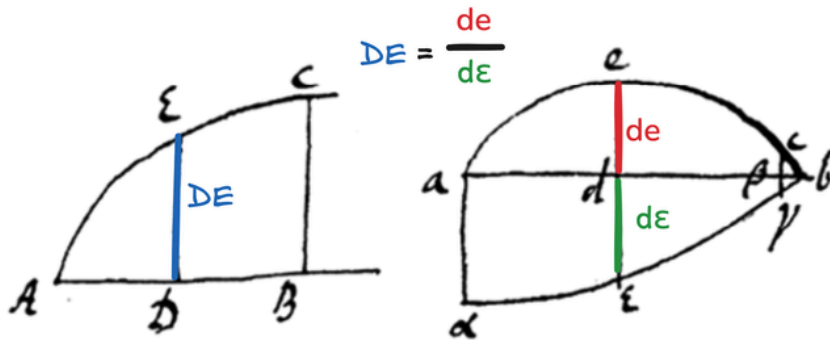
Bernoulli explains L'Hôpital's rule to L'Hôpital (iii)

Therefore, to find the magnitude of the ordinate BC , I construct on the same axis adb two other curves aeb and $\alpha\epsilon b$ of such a nature that having taken abscissas equal to AD and ad , the ordinates de are in ratio to the numerator of the general fraction, which expresses the ordinate DE , and $d\epsilon$ are in ratio to the denominator of the same fraction.



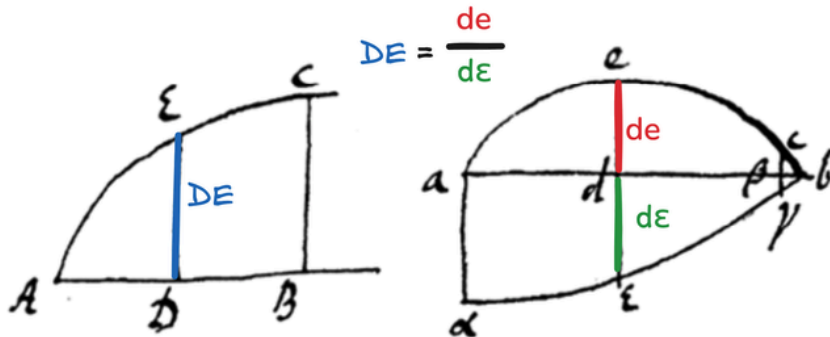
Bernoulli explains L'Hôpital's rule to L'Hôpital (iv)

This being done it is clear that de divided by $d\epsilon$ may be supposed equal to DE . The problem therefore reduces to finding the value of de divided by $d\epsilon$ in the case that ab is equal to AB .



Bernoulli explains L'Hôpital's rule to L'Hôpital (v)

Now, I see that in this case, de and $d\epsilon$ vanish because the two terms of the fraction vanish, and thus the two curves aeb and $\alpha\epsilon b$ intersect at the point b .

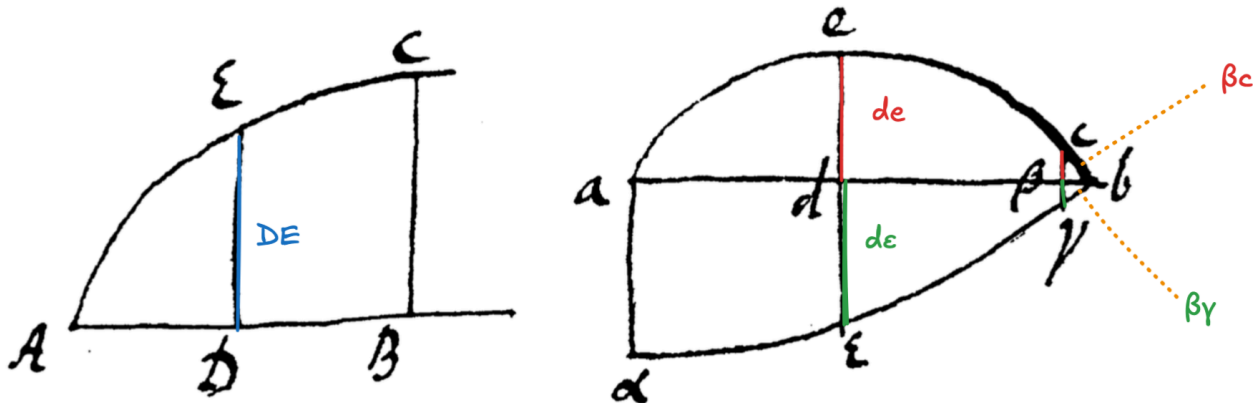


$$DE = \frac{de}{d\epsilon}$$

Bernoulli explains L'Hôpital's rule to L'Hôpital (vi)

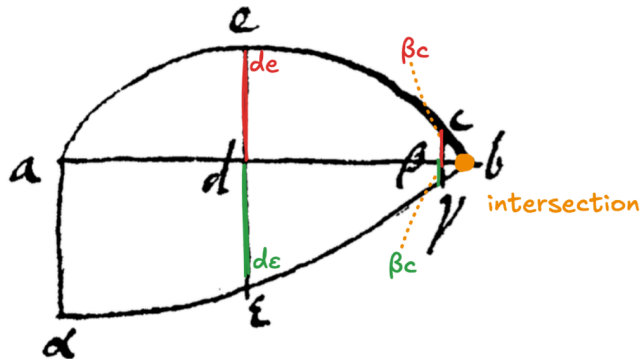
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Therefore, we need only take the last differentials βc and $\beta \gamma$, of which the one divided by the other will tell me the magnitude of BC that I seek



Bernoulli explains L'Hôpital's rule to L'Hôpital (vii)

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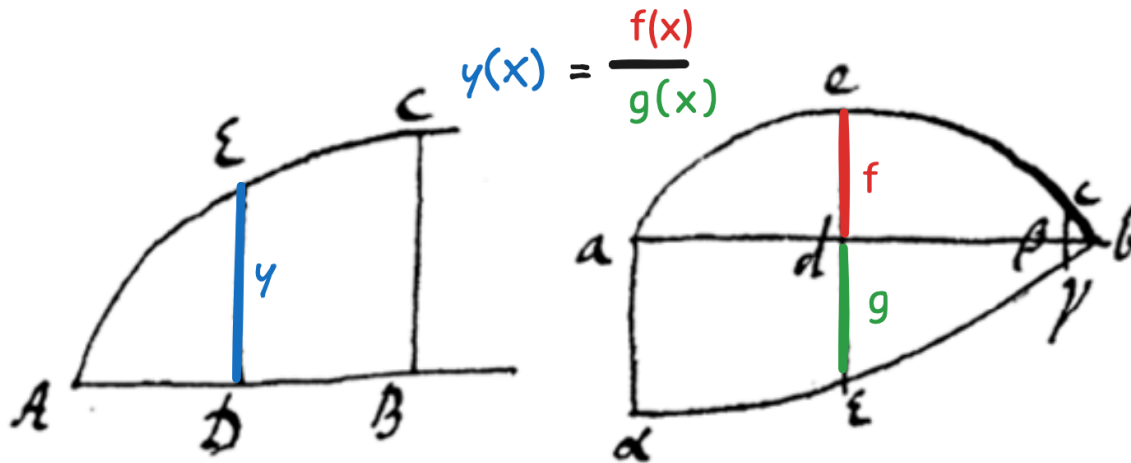
Curves intersect $\Rightarrow \frac{de}{d\varepsilon} = \frac{\beta c}{\beta\gamma}$ at b?

Postulate I

Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

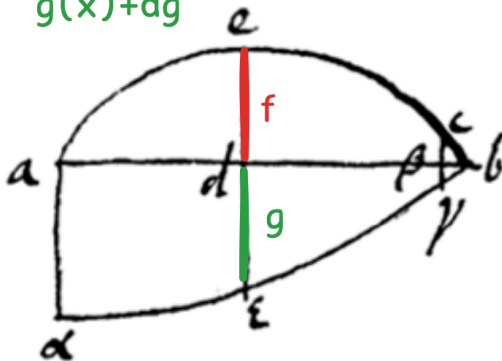
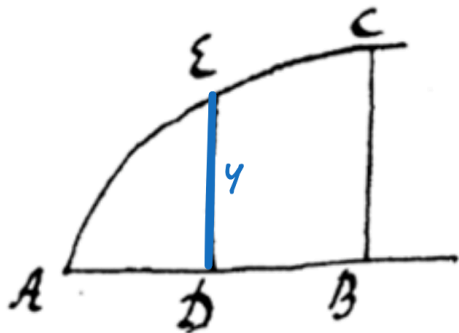
$x \rightarrow dx + x$ is for the purposes of calculation

Bernoulli explains L'Hôpital's rule to L'Hôpital (viii)



Bernoulli explains L'Hôpital's rule to L'Hôpital (ix)

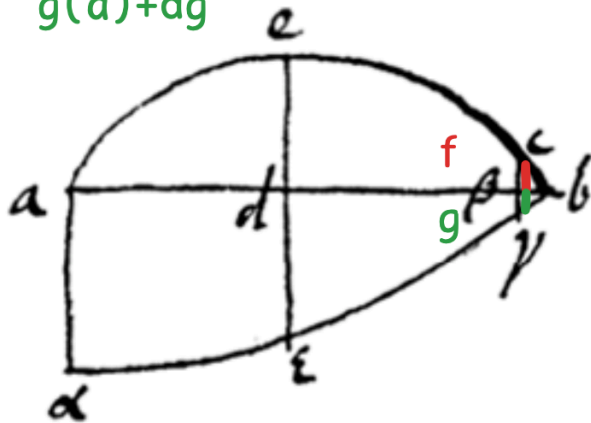
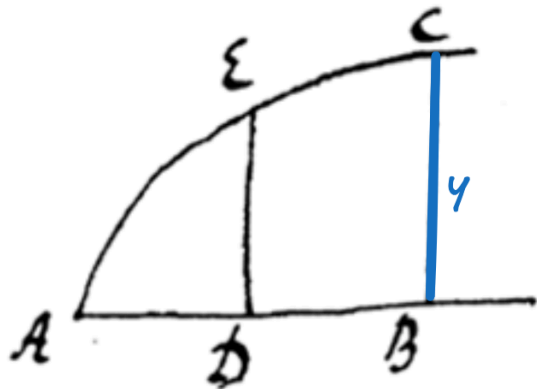
$$y(x) = \frac{f(x) + df}{g(x) + dg} \quad \text{Postulate 1}$$



Bernoulli explains L'Hôpital's rule to L'Hôpital (x)

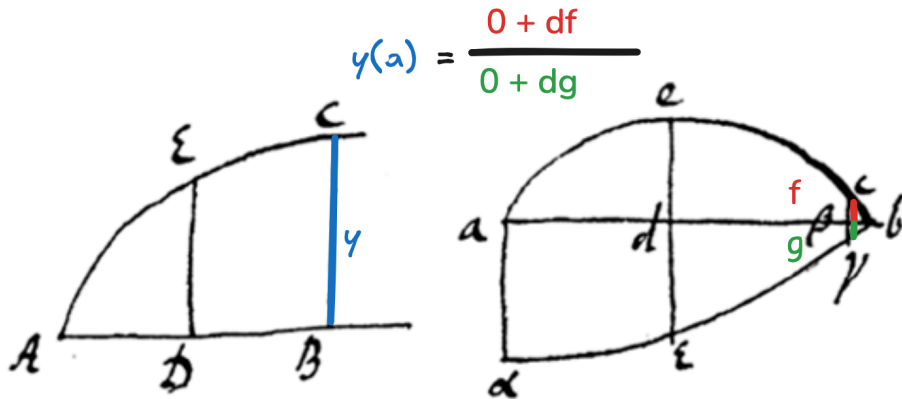
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$$y(a) = \frac{f(a) + df}{g(a) + dg}$$



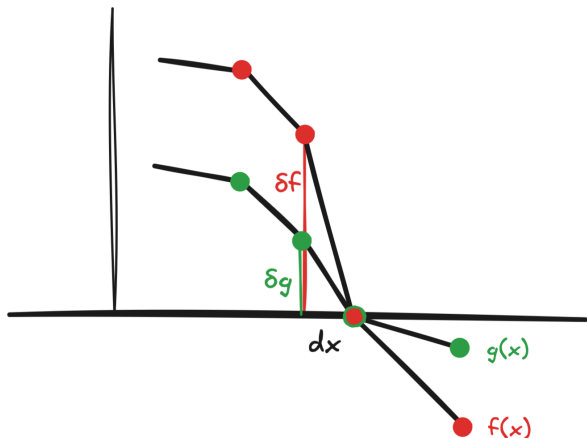
Bernoulli explains L'Hôpital's rule to L'Hôpital (xi)

Curves intersect:



Summary of Argument

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Postulate II

Curves are made of infinitely many infinitesimal line segments.

Postulate I

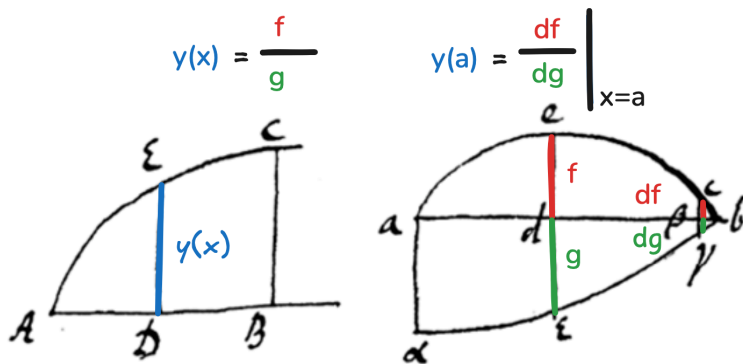
Quantities that change by an infinitesimal remain the same

PI + PII + Curves Intersect and are both 0

$\rightarrow \frac{df}{dg} = \frac{f}{g}$ at the intersection point

Full statement of the Rule (July 1694)

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This is what gives me the following general rule: *To find the value of the ordinate of the given curve in the given case we must divide the differential of the numerator of the general fraction by the differential of the denominator; the quotient, after having made x equal to the supposed AB , will be the magnitude of BC*

Example. The curve ACE has for its equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} = y.$$

Thus, if AB is $= a$, we have $BC = \frac{0a}{0}$, now we wish to know the true value.

Example Problem (ii)

According to the rule, I take the differential of the numerator $\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}$, which is

$$= \frac{a^3 dx - 2x^3 dx}{\sqrt{2a^3x - x^4}} - \frac{a^2 dx}{3\sqrt[3]{aax}}$$

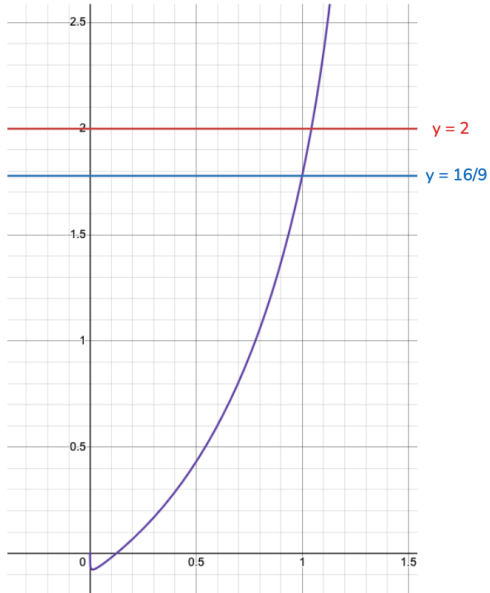
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Example Problem (iii)

having now substituted in the place of x the supposed value a , we find $-\frac{4}{3}a dx$ for the first differential and $-\frac{3}{4} dx$ for the second one. Therefore,

$$\frac{-\frac{4}{3}a dx}{-\frac{3}{4} dx} \quad \text{or} \quad \frac{16a}{9} = BC.$$

Example Problem (iv)



Let $a = 1$

$$y = \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - \sqrt[4]{x^3}}$$

1696:

ANALYSE
DES
INFINIMENT PETITS,

POUR
L'INTELLIGENCE DES LIGNES COURBES,

Par M^r le Marquis DE L'HOSPITAL.

SECONDE EDITION.

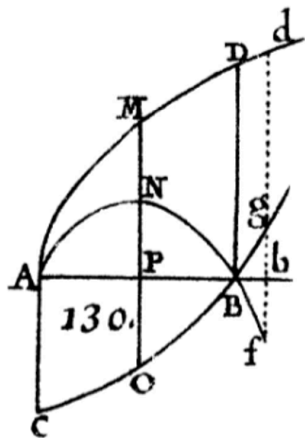


A PARIS,

Chez FRANÇOIS MONTALANT, Quay des Augulins,

M D C C X V.

AVEC APPROBATION ET PRIVILEGE DU ROY.



[...] if we imagine an ordinate bd infinitely close to BD , which meets the curved lines ANB and COB at f and g , then we will have $bd = \frac{AB \times bf}{bg}$, which (see Postulate 1) does not differ from BD . It is therefore only a question of finding the ratio of bg to bf

Postulate I

Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

- Leibniz' early papers were not widely understood
- The *Analyse* (1696) was the first calculus textbook: popular, widely read in France throughout the 17th century⁸
- Believed to be original work until early twentieth century.

the Geometry of the Infinitely small was still nothing but a kind of Mystery, and, so to speak, a Cabalistic Science shared among five or six people. They often gave their Solutions in the Journals without revealing the Method that produced them, and even when one could discover it, it was only a few feeble rays of this Science that had escaped, and the clouds immediately closed again.

— Fontenelle, 1708

⁸Truesdell, "The New Bernoulli Edition".

Bernoulli complains to Varignon:

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to speak frankly, Mr. de L'Hôpital had no other part in the production of this book than to have translated into French the material that I gave him, for the most part, in Latin

— Bernoulli to Varignon, 1707

	L'Hôpital's Theorem 2026	L'Hôpital's Theorem 1696
... is about	The limit of a quotient of two functions $\frac{f(x)}{g(x)}$	The actual value of the quotient.
...uses	derivatives defined via limits : $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$	ratios of differentials (dy, dx) which belong to the infinitesimal straight-line segments which make up all curves
... is proved using	The Mean Value Theorem (real analysis).	Postulates 1 and 2 of the <i>Lectiones</i> (1692) ⁹

⁹Bradley et al., "Bernoulli's Lectiones De Calculo Differentialis".

- Bradley, Robert E., Salvatore J. Petrilli, and C. Edward Sandifer. “Bernoulli's Lectiones De Calculo Differentialis.” In *L'hôpital's Analyse Des Infiniments Petits: An Annotated Translation with Source Material by Johann Bernoulli*. Springer International Publishing, 2015. https://doi.org/10.1007/978-3-319-17115-9_11.
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