Blue Text is for comments which will be deleted before we hand in the final version

To modify Victor's illustrations you can here (download and open in excalidraw)

DRAFT
P3 -group 7

Isaac Newton's Power Series for Sine and Cosine

Emma Ottenhof and Victor Elgersma

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Contents

1. Introduction

We provide a line-by-line commentary on sections §37-§47 of Newton's *Of Analysis by means of Equations with an infinite number of terms* [1, pp. 335-339], first published in Latin in 1711. Newton writes that he made many of these discoveries in 1665-1666 [2, §2].

We feel that Newton's diagrams suffer from the use of too many letters, distracting the modern reader from the clarity of his argument. We will use letters sparingly in our interpretation, and rely instead on colours instead.

1.1. How to understand "Moments"

E: I'm a little unsure about Newton's meaning of moment... In one of the modern texts it says that Newton considers the moment of the arc αD to be DH, i.e. dz, and the moment of the base AB the part BK, i.e. dx (these letters referring to Newton's second figure). So that made me think the moment in his view is actually the infinitesimal increment of a quantity. V: Yes, I think we should proceed by treating Moments as infinitesimal line segments rather than derivatives. Essentially because derivatives are what Newton calls Fluxions. V: This interpretation also explains why Newton does not mention Fluxions in this passage. Since Fluxions are always numerically equal (but dimensionally different from) the Moments, it may be confusing to explicitly mention them in the same passage.

For our glossary, we have taken the definition of "Moment" from Robert Pyke [3], but this is not so evident from a first glance at Newton's text. If a Moment is fluxion × time, as Pyke would have it, where are all the dts (or equivalently - the small quantities *o*) in the objects Newton calls Moments?

Let's take a few examples. In §37 Newton writes that the area x (Figure 1) is described by the Moment "1". To our modern eyes this "1" might be most readily understood as the derivative w.r.t t of $x = t$. But if this really were the case, why call this a Moment and not a Fluxion? Given that Newton has reserved this special technical term for the Fluxion, it is more reasonable to understand this Moment "1" as "$1dt$", where the little increment of time is implicit.

Another example: in §38 we appear faced with an apparent contradiction. First, the "Moment of the Arch AD" is supposed to be the infinitesimal line segment HD (Figure 1). Next, the *very same moment* is designated $\frac{\sqrt{x-x^2}}{2x-x^2}$, which looks a lot like an expression of the *derivative* of the arch length with respect to the x coordinate, $\frac{dz}{dx}$.

TODO: Add clearer diagram for what dz is here.

What is going on? Do moments have a dual nature, where they are sometimes derivatives and sometimes line segments? Our best guess is that, like the $BK = 1$ example above, Newton leaves out the small increment of *time* implicit in his Moment definitions, so we should really understand his moment as being $\frac{\sqrt{x-x^2}}{2x-x^2}dt$ - which is equal to $\frac{\sqrt{x-x^2}}{2x-x^2}dx$ when $x = t$.

In fact, this last equality, $\frac{\sqrt{x-x^2}}{2x-x^2}dt = \frac{\sqrt{x-x^2}}{2x-x^2}dx$, follows from $x = t$. Newton *always* assumes a "uniform" motion of x, and since in §37 we find BK=1, $x = t$ follows. And it is this last expression that Newton justifies in §40, the final red herring when it comes to understand Moments. At first glance, §40 appears to suggest that Moments are a derivative - because they always *have a dimensionality one less than the form they generate*. If Moments were line (or area, or volume) segments, then surely in §40 Newton would say that the moments would have the *same* dimensionality as the curves they generate. But, like in §37, if we assume that there is always an implicit dt next to the "Unity" that Newton "puts for the Moment", the contradiction disappears. Time t and distance x have the same dimensionality for Newton. Therefore, an implicit dt multiplying every moment will give us the dimensionality allowing us to view moments as infinitesimals, rather than derivatives.

2. Line by line analysis of Newton sine series

V: think it is good to separate out each section of Newton's text with its own subheading - i.e something like `"== sec[92]"`

> *The Application of what has been said to other Problems of that Kind.*

2.1. §37 The Moments at All Times allow us to recover the Quantities

In this section, Newton illustrates his claim that knowing the Moments of a Quantity (or 'fluent') at all times allows you to calculate that Quantity at all times as well.

> 37. Let ABD be any Curve, and AHKB a Rectangle, whoſe Side AH or BK is Unity :

$AHKB$ is a two-dimensional x-axis whose side length is 1 (see Figure 1). When considering areas under curves Newton prefers to consider a 2-dimensional x-axis, as opposed to a one-dimensional x-axis.

> **? Conceptual Question**
>
> We also saw a two-dimensional x axis this in Newton's Treatise of the Quadrature of Curves in presentation P1-8. Can we say something about the comparison of the two?
>
> Maybe we can relate this to [2, §20]. (Since all derivatives are time derivatives, setting $\dot{x} = 1$ and all higher derivatives of x to zero means all time-derivatives become x derivatives.)

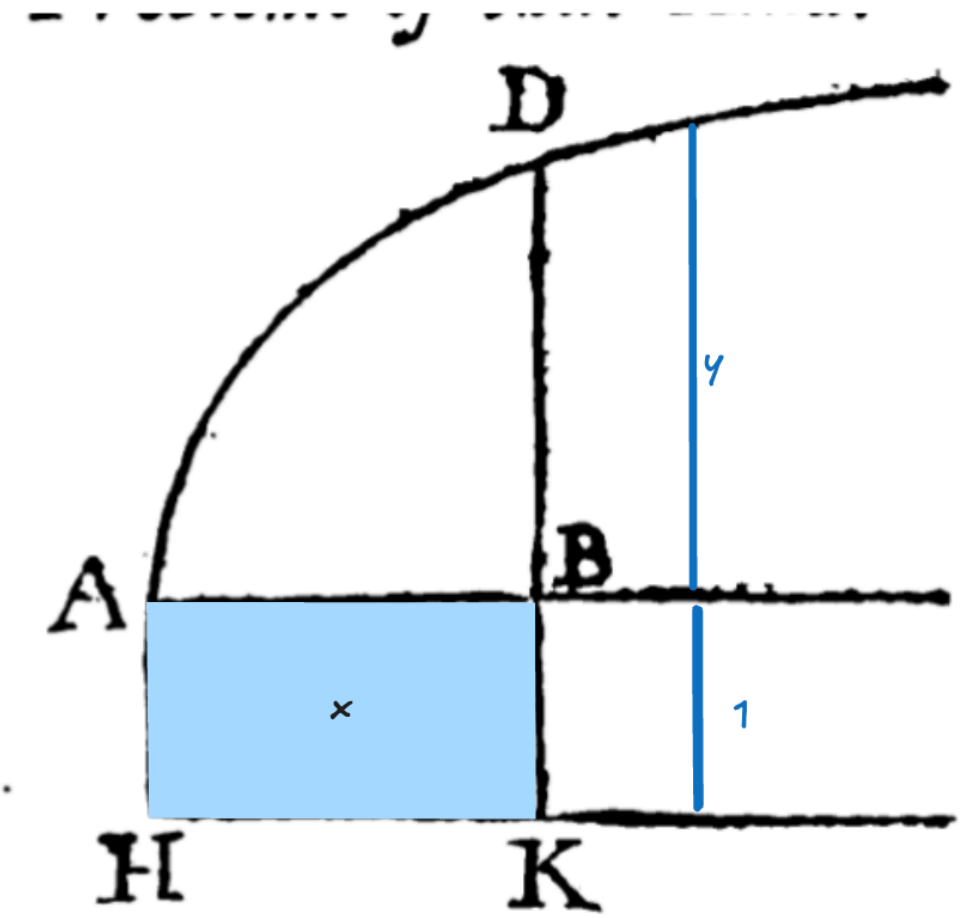


Figure 1: The blue area (x) is "increased continually by the Moment 1". We would understand the "Moment 1" as simply dx but Newton does not use this notation, instead thinking of the moment as $\dot{x}o$ where o is a small quantity which can be omitted as it is always taken to 0 at the end of the procedure Newton describes in [2, §17], and in this case $\dot{x} = 1$. In §20 Newton explains why it is sometimes useful to pick a quantity like x, set $\dot{x} = 1$ and make all other fluxions vanish.

As we can see from Figure 1, there is an obvious parallel between increasing the area of the rectangle by the 'Moment' 1 and increasing the area of the arbitrary curve by the 'Moment' y.

V: Although this doesn't explain why Newton didn't simply put these two examples on different diagrams - why not have an area $z(x)$ by swept out by the line $y(x)$ dependent on a one-dimensional x axis? Why two-dimensional?

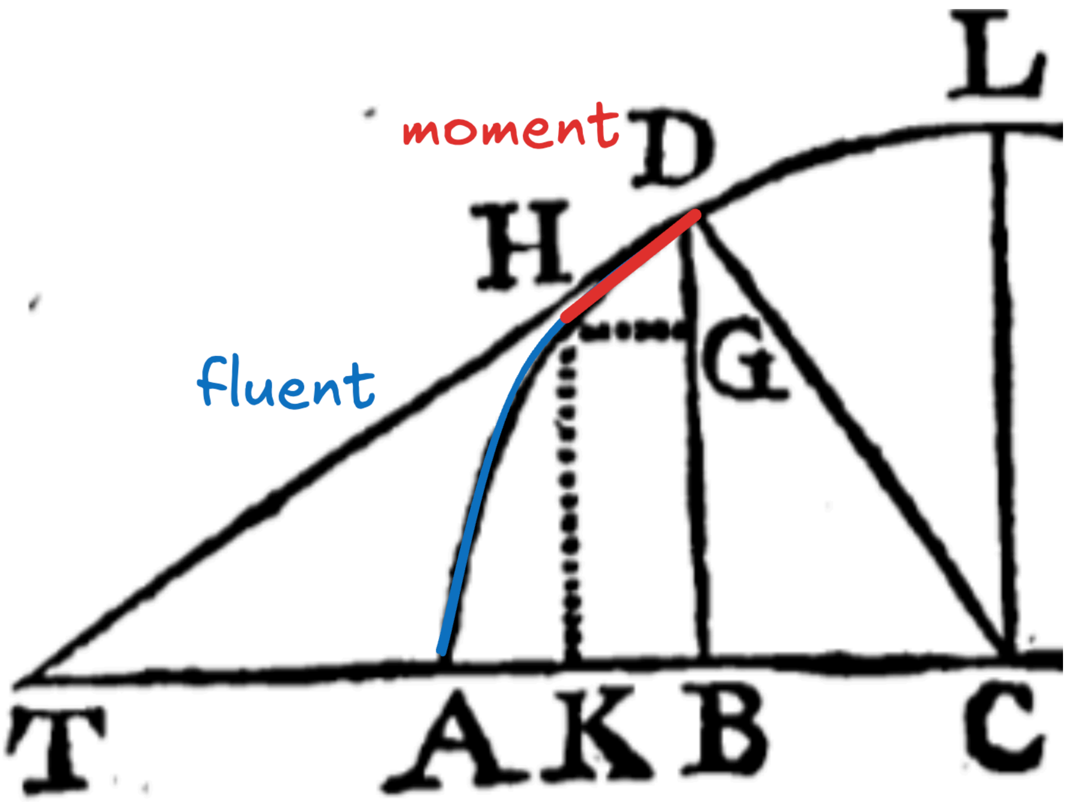


Figure 2: §38 - Newton considers the arc length (blue) to be generated by the moment (red)

> And imagine the Right Line DBK to move uniformly from AH, ſo as to deſcribe the Areas ABD and AK; and that BK (1) is the

Newton's calculus is interesting because, on the one hand it puts Time at the center of everything (see Appendix A. 1), but it never includes time t as a parameter. Perhaps it will be illustrative to make the time-dependence explicit. Newton's 'uniform' motion can be satisfied by letting:

$$x = t \tag{1}$$

Then, $x = t \Rightarrow \dot{x} = 1$, and thus the Fluxion of x is 1 and the Moment BK is $1dt$ (or $1dx$). Since this dt would feature in every moment, Newton leaves it out[1]. Newton gives his "1" the dimensionality of a line. This would seem to cast doubt on identifying the moment with the differential area segment, because it is the *derivative* of the area x that has the dimensionality of a line, not the differential area segment dx, but we understand it as follows. If we assume that time and length have the same dimension, then the "1" in the differential area segment $1dt$ will indeed have the dimensionality of a line. So, to conclude, whenever Newton speaks of the "moment" f, we should think of it as an infinitesimal segment of the fluent in question, fdt, but with the proviso that Newton is working with units in which time and length have the same dimensionality

[1] Not only does he leave out the dt, he gives the "Moment" the dimensionality of what we would understand as the derivative - see §40

> "Areas ABD and AK; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreaſed; and that from the Moment BD

So x is "gradually increased" by $1dt$ (equivalently, $1dx$, or the line of length 1) and y is "gradually increased" by ydt (equivalently, ydx, or the line of length y)

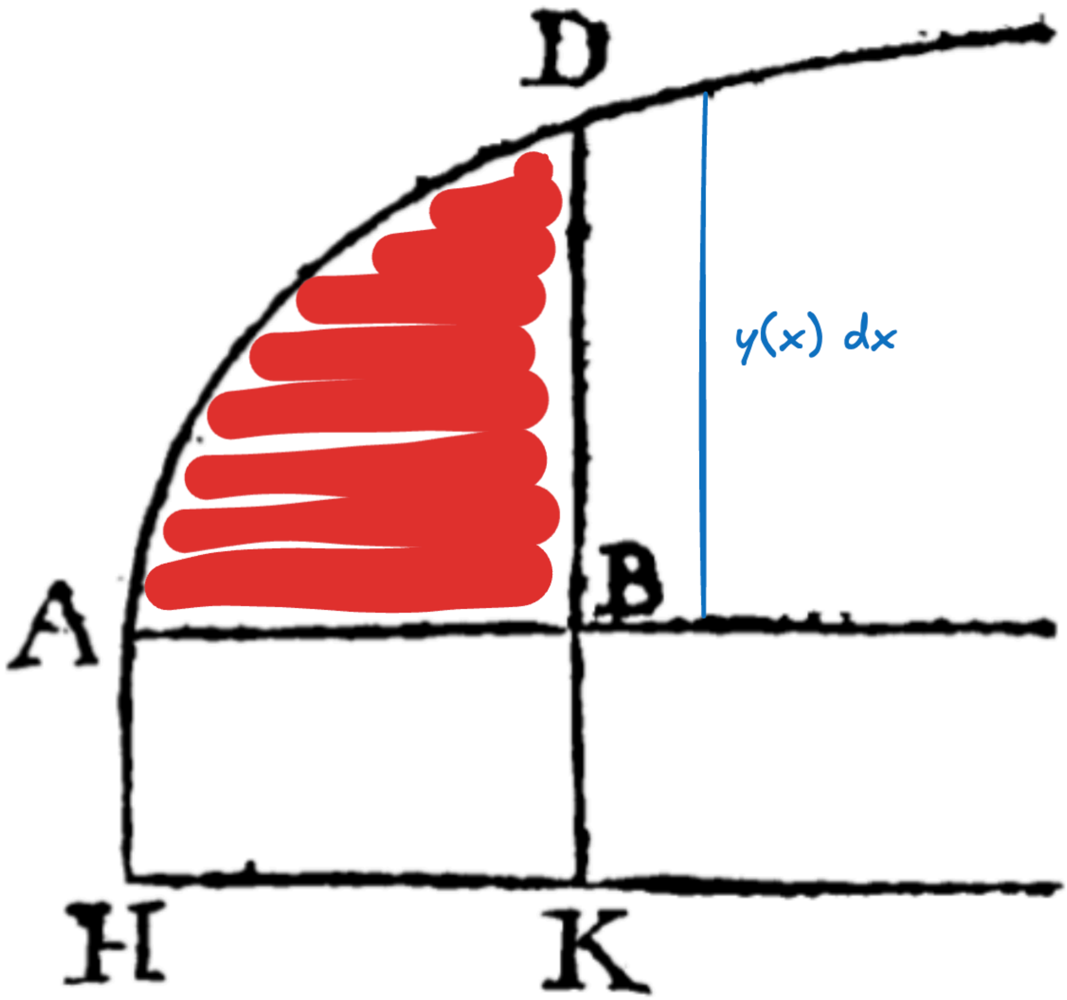


Figure 3: the red area is increased continually by the Moment $y(x)$. Equivalently, it is increased by the infinitesimal area segment $y(x)dx$

> encreased; and that from the Moment BD continually given, you can, by Means of the preceding Rules, investigate the Area ABD described by it, or compare it with AK (x), which is decribed with the Moment 1.

AK is really shorthand for the area $ABKH$. When Newton says "investigate the Area ABD described by it, or *compare* it with $AK(x)$", we read him as drawing attention to how the area under the curve changes *as a function* of x. This may be why Newton wants x to be an Area, for comparing areas makes more intuitive sense than comparing Areas to lines.

In his *Treatise on the Quadrature of Curves*, Newton says that it is useful, sometimes to take a quantity like x, set its first fluxion equal to Unity and ignore all higher Fluxions. In this way, time-derivatives become x-derivatives [2, §20, p. 7], and we can use the inverse fluxion method to integrate lengths.

> Now by the same Means that the Superficies ABD from it's Moment being at all Times given, is discovered by the foregoing Rules, by the like Means may any other Quantity be investigated from it's Moment given in like manner. The Thing will be clearer by an Example.

In this passage, "Quantity" should be understood as a Fluent, i.e varying with time. If we know the Moment ($y(x)dt$) of a Fluent at all times, we can calculate the Superficies ABD (the integral $\int y(x)dx$). It is striking that Newton still uses an x instead of a t for his independent variable, as it is clear from this discussion that the Moment should be viewed as a function of time. Indeed, since x is supposed to vary uniformly with time, knowing y "at all times" is equivalent to knowing the function $y(x)$. "any Quantity may be investigated from it's Moment" is equivalent to saying - "Any Moment (derivative) can be integrated".

> **? Conceptual Question**
>
> In light of [2, §17, p. 6], Why does Newton write 1 for the moment instead of $1o$?
>
> Answer:
>
> I think it has to do with §1, in which he explicitly denies that curves are made of infinitesimal line segments. Writing moments as having a definite extent is too close to the view that he is arguing against. Therefore he always leaves out the os

2.2. §38 Finding the arc length of a circle in terms of a coordinate x

In this section, Newton uses his method of *inverse fluxions* to find an arbitrary arc length of a circle in terms of a coordinate on its diameter:

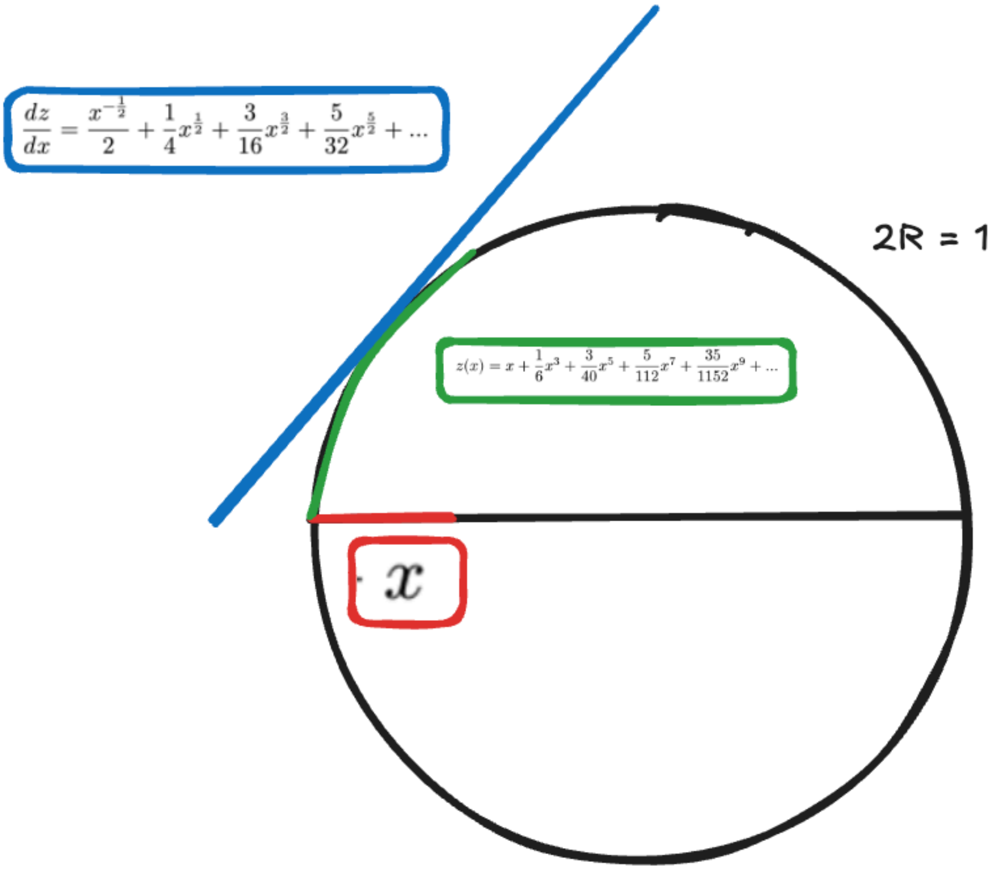
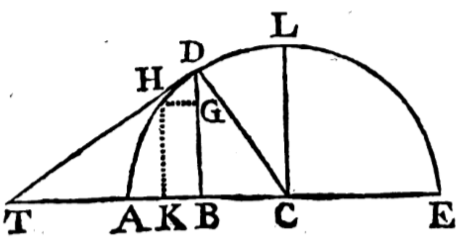


Figure 4: Summary of §38. Circle has diameter of 1

> *To find the Lengths of Curves.*
>
> 38. Let ADLE be a Circle, the Length of whofe Arch AD is to be inveftigated. Draw the Tangent DHT, and having compleated the indefinitely fmall Rectangle HGBK, and put AE = 1 = 2AC, it fhall be as BK or GH the Moment of the Bafe AB (x) to HD the Moment of the Arch AD : : BT : DT : : BD ($\sqrt{x - xx}$) : DC ($\frac{1}{2}$) : : 1 (BK) : $\frac{1}{2\sqrt{x - xx}}$ (DH). And fo
>
> 
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Interestingly, whereas §38 is given as concrete example of §37, the dimensionalities are different. §38 involves integrating an ordinate $y(x)$ to find an area, whereas in §38 we integrate over infinitesimal line segments to find an arc length. Nevertheless, Newton considers these examples close enough that one is given as an example of the other.

We find it puzzling that Newton insists on creating a $\frac{1}{2}$-unit circle rather than a unit circle at this point, especially considering that in §39 he considers a unit circle.

❓ Conceptual Question

Do you get a different answer if you consider a unit circle at this point?

When Newton writes "BD ($\sqrt{x - xx}$) : DC $\left(\frac{1}{2}\right)$", the brackets should not be taken to mean multiplication. Instead, the statement "$A(B) : C(D)$" actually means "$\frac{A}{C} = \frac{B}{D}$". E: Yes I suppose that's true although I tend to think more simply about these brackets as indicating equality, like he's saying "A (which by the way is equal to B) stands to C (which by the way is equal to D)..." but of course that has the same implications as what you describe. For example, "$1(BK) : \frac{1}{2\sqrt{x-x^2}}(DH)$" actually means "$\frac{BK}{DH} = 2\sqrt{x - x^2}$".

We start by noticing that the red and green triangles are similar (see Figure 5) since the two triangles share the angle $\angle BDT$ (or $\angle GDH$) and both triangles contain a right angle ($\angle TBD$ and $\angle HGD$), whence:

$$\frac{DT}{BT} = \frac{DH}{GH} \tag{2}$$

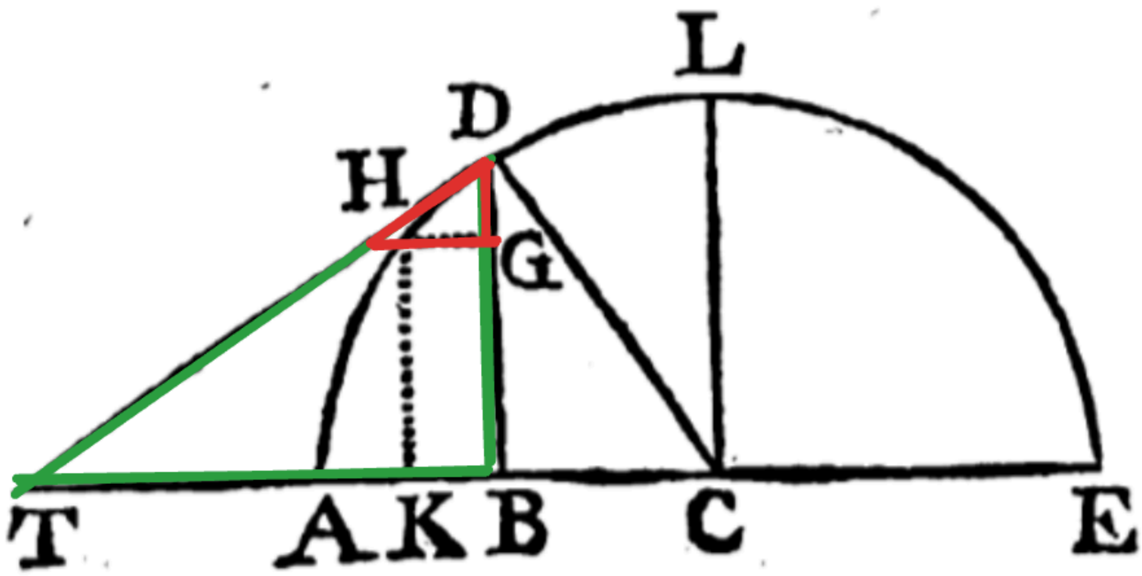


Figure 5: red and green triangles are similar

Next, we notice that the red and green triangles in Figure 6 are also similar, since both contain a right angle ($\angle TBD$ and $\angle DBC$) and $\angle CDB = \angle BTD$. The latter follows from the fact that in $\triangle DBT$, we can see that $90° - \angle BDT = \angle BTD$, and since $\angle CDT = 90°$, we know that $\angle CDB = 90° - \angle BDT$, which we established was equal to $\angle BTD$. TODO: reword for clarity

Therefore:

$$\frac{DT}{BT} = \frac{DC}{BD} \tag{3}$$

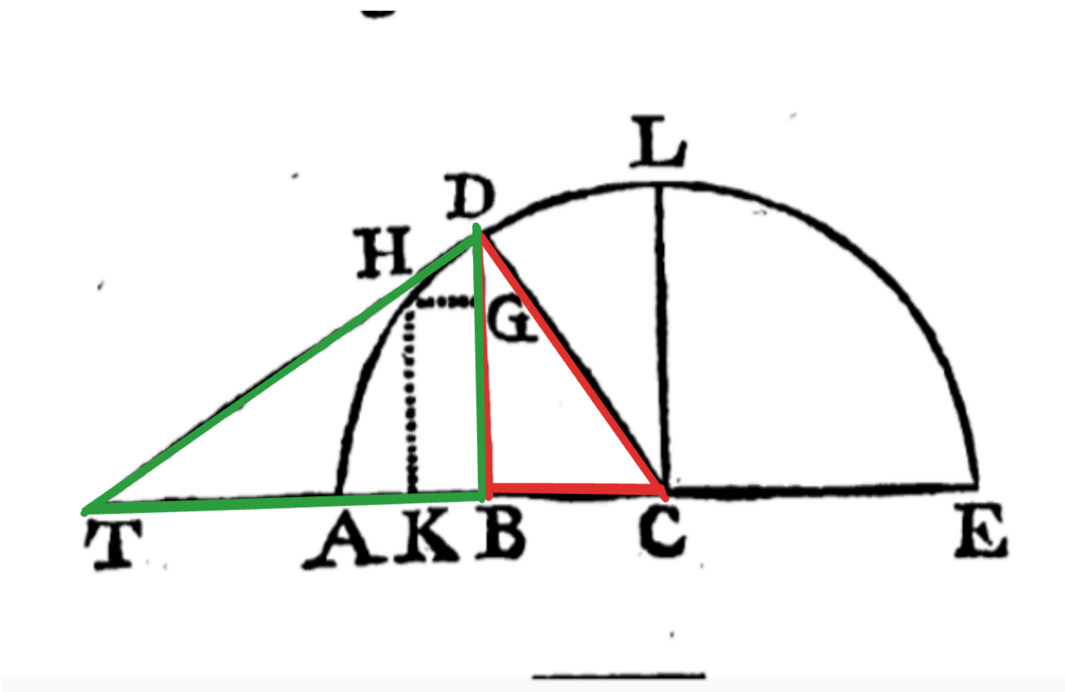


Figure 6: red and green triangles are similar

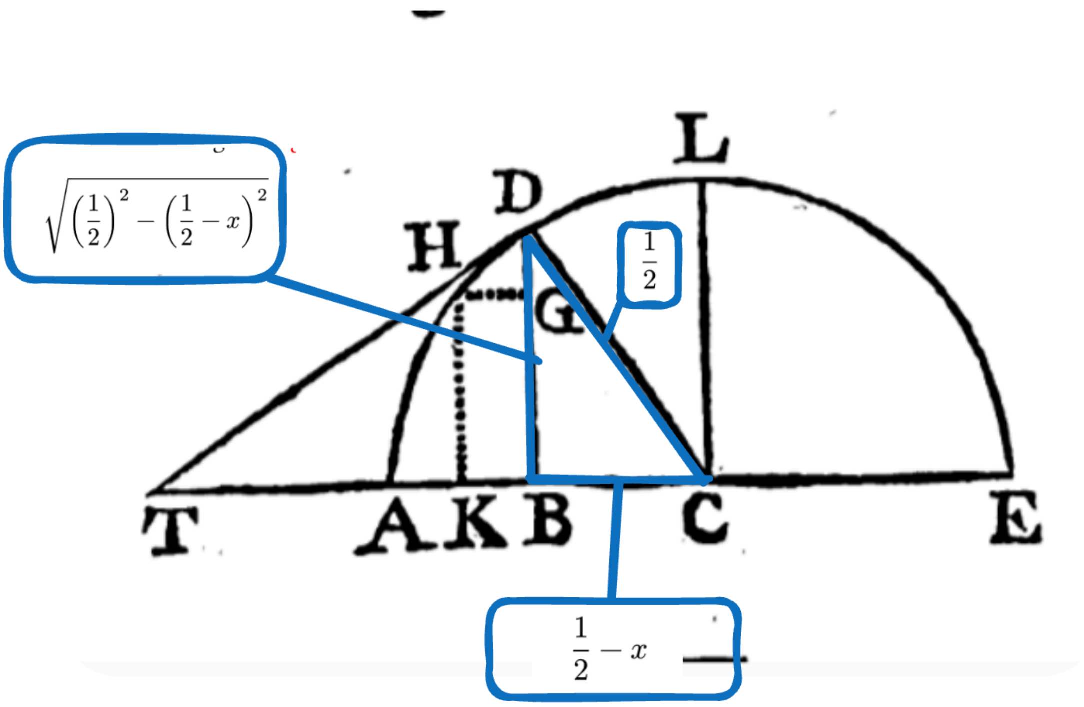


Figure 7: The Pythagorean theorem is used to find the value of the line BD

Next, we use the pythagorean theorem on the blue triangle in Figure 7, to find that:

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} = \sqrt{x - x^2} \tag{4}$$

Finally, by constructing the circle to have a radius of $\frac{1}{2}$, we know that:

$$DC = \frac{1}{2} \tag{5}$$

Combining Equation 2 and Equation 3 to eliminate $\frac{DT}{BT}$ gives:

$$\frac{DH}{GH} = \frac{DC}{BD} \tag{6}$$

Using Equation 4 and Equation 5 to substitute for DC and BD gives:

$$\frac{DH}{GH} = \frac{1}{2\sqrt{x - x^2}} \tag{7}$$

Now we can rewrite our infinitesimal triangle in a way that will be more recognizable to modern readers, expressing the variable length of the arc AD as $z(x)$: TODO: illustrate d x, d a on a diagram

$$\frac{DH}{GH} = \frac{dz}{dx} \tag{8}$$

Therefore:

$$\begin{aligned} \frac{dz}{dx} &= \frac{1}{2\sqrt{x - x^2}} \\ z(x) &= \int \left(\frac{1}{2\sqrt{x - x^2}} \right) dx \end{aligned} \tag{9}$$

In Newton's words, $\frac{1}{2\sqrt{x-x^2}}$ is the 'Moment' - i.e *derivative* - of the arc AD (or $z(x)$ for us).

We can perform the integral by first writing out the series expansion for $\frac{1}{2\sqrt{x-x^2}}$, then integrating term-by-term.

The binomial theorem is easier to apply if we first rewrite the Moment as:

$$\frac{1}{2\sqrt{x - x^2}} = \frac{1}{2}(x - x^2)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2}(1 - x)^{-\frac{1}{2}} \tag{10}$$

Next we expand $(1 - x)^{-\frac{1}{2}}$ using Newton's recursive version of the Binomial theorem, namely:

$$(P + PQ)^{\frac{m}{n}} = A + B + C + D + E + ... \tag{11}$$

where:

$$\begin{aligned} A &= P^{\frac{m}{n}} \\ B &= \frac{m}{n} AQ \\ C &= \frac{m - n}{2n} BQ \\ D &= \frac{m - 2n}{3n} CQ \\ &\text{etc...} \end{aligned} \tag{12}$$

Where $P = 1$ and $Q = -x$, $m = -1$ and $n = 2$, we have:

$$
\begin{aligned}
A &= 1 \\
B &= \frac{1}{2}x \\
C &= -\frac{3}{4}\left(\frac{1}{2}x\right)x = -\frac{3}{8}x^2 \\
D &= \frac{-1-2(2)}{3\times 2}\left(\frac{3}{8}x^2\right)x = \frac{5}{16}x^3
\end{aligned} \tag{13}
$$

So:

V: we should probably add all of N.'s terms

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \tag{14}$$

Therefore:

$$\frac{1}{2\sqrt{x-x^2}} = \frac{x^{-\frac{1}{2}}}{2}(1-x)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \dots \tag{15}$$

Now that we have obtained an expression for the moment of arc $\frac{dz}{dx}$ as a function of x (see Figure 8), we can integrate it term by term to get the arc length

$$
\begin{aligned}
\frac{dz}{dx} &= \frac{x^{-\frac{1}{2}}}{2} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \dots \\
z(x) &= \int z(x)dx = x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}\dots
\end{aligned} \tag{16}
$$

Which is

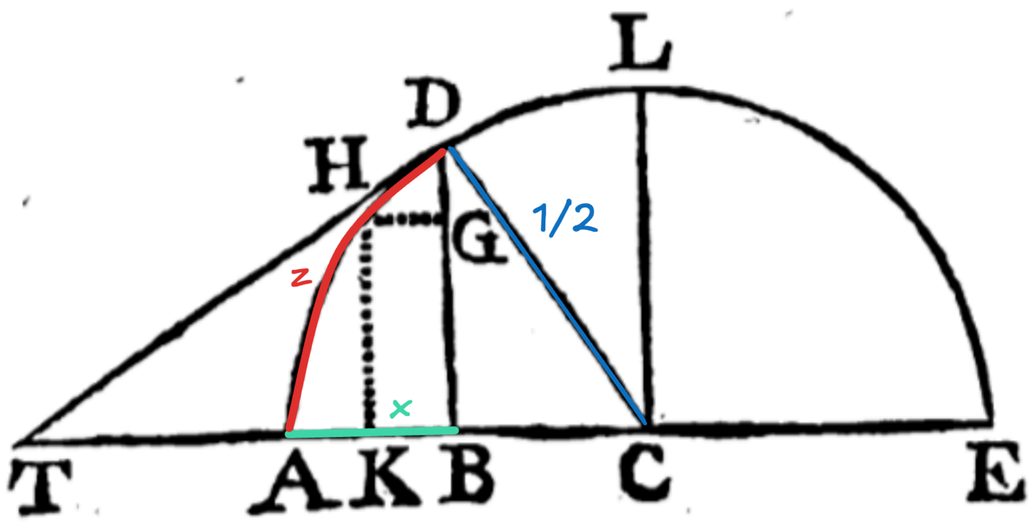


Figure 8: Finding $\frac{dz}{dx}$

2.3. §39 Finding the arc sine

> 39. After the ſame Manner by ſuppoſing CB to be x, the Radius CA to be 1, you will find the Arch LD to be $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

We now use the inverse fluxional method to find a power series expansion of the arc sine, that is we write the arc length z as a function of the base of the right triangle x in Figure 9.

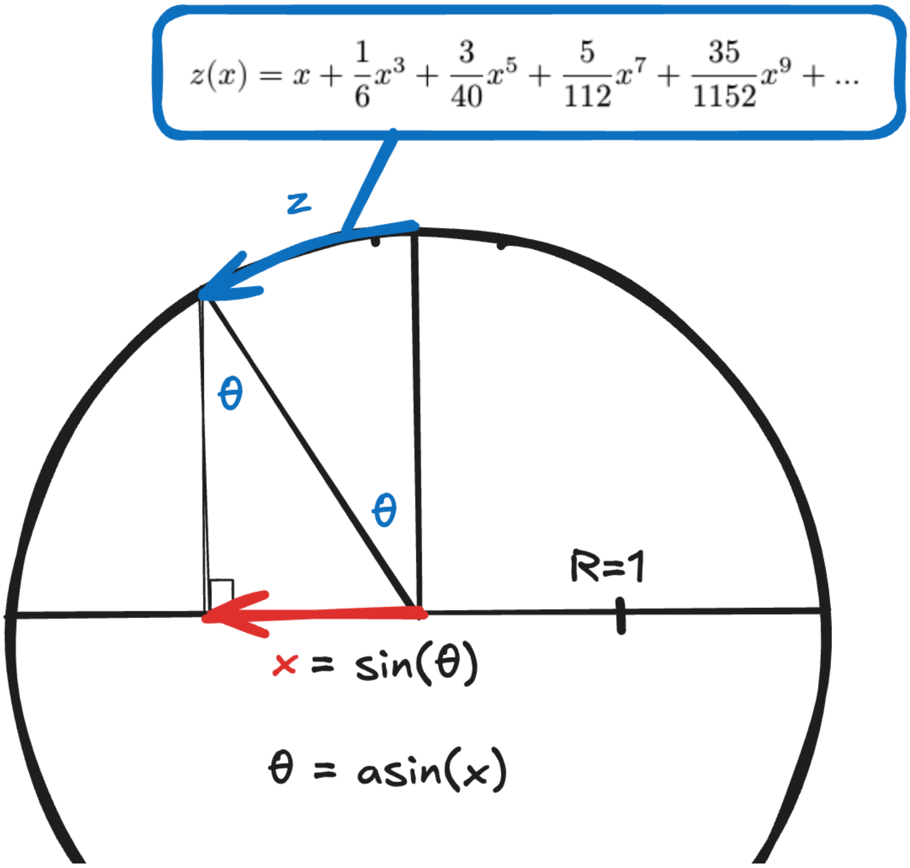


Figure 9: Summary of §39, compare with Figure 4

In contrast with §38, we now use a unit circle rather than a circle with diameter 1, so:

$$DC = 1 \tag{17}$$

and instead of defining $AB = x$ we have $BC = x$. By Pythagoras' theorem we obtain

$$BD = \sqrt{1 - x^2}. \tag{18}$$

Combining Equation 6 and Equation 17 we find that

$$\frac{dz}{dx} = \frac{1}{\sqrt{1 - x^2}} \tag{19}$$

which gives

$$z(x) = \int \left(\frac{1}{\sqrt{1-x^2}} \right) dx. \tag{20}$$

V: Can use Newton's own theorem here as. well.

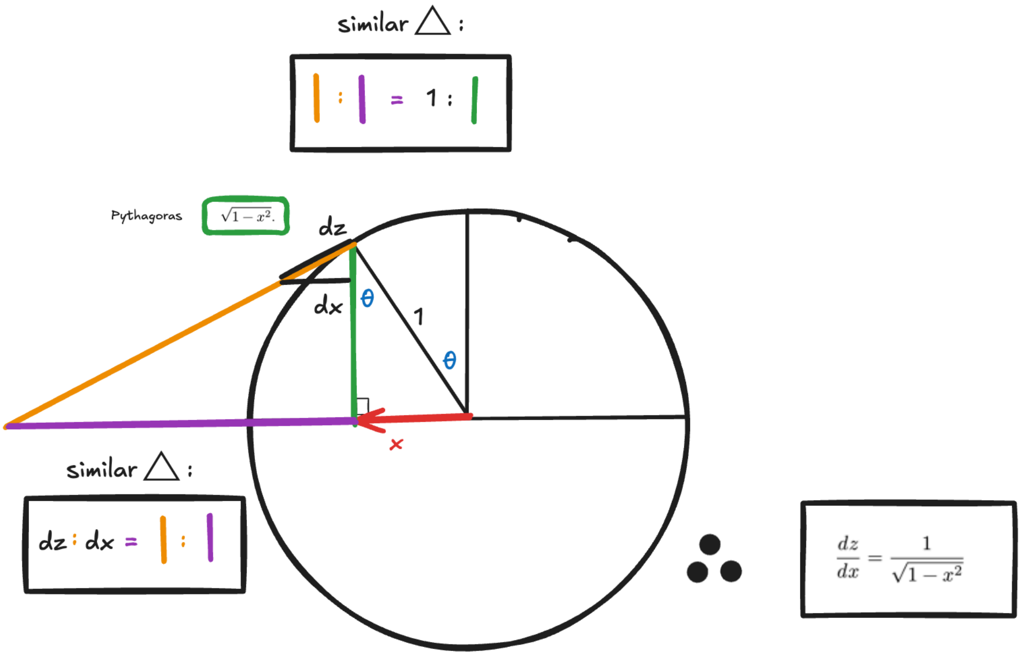


Figure 10: a similar triangles proof without letters.

As before, applying a series expansion to the integrand allows us to re-write this expression as

$$z(x) = \int \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \ldots \right) dx \tag{21}$$

and integrating term by term gives

$$z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots \tag{22}$$

2.4. §40 Dimensionality of Unity

> 40. But it is to be remarked that that Unity which is put for the Moment, is a Superficies, when the Queſtion is about Solids; and a Line when about Superficies; and a Point when it is about Lines (as in this Example.) Neither am I afraid to ſpeak of Unity in Points, or Lines infinitely ſmall, ſince Geometers are wont now to conſider Proportions even in ſuch a Caſe, when they make uſe of the Methods of Indiviſibles.

2.5. §41 Solids and Centers of Gravity.

> 41. From theſe Things one may gueſs how one ought to proceed in inveſtigating the Superficies and Contents of Solids; and likewiſe the Centers of Gravity.

V: Can we say anything about centers of gravity?

V: If a 2D x axis is needed to allow us to compare the area $\int y(x) dx$ with x in §37, would we need a cylindrical x-axis with area 1 to allow us to compare the volume of rotation of a curve against x

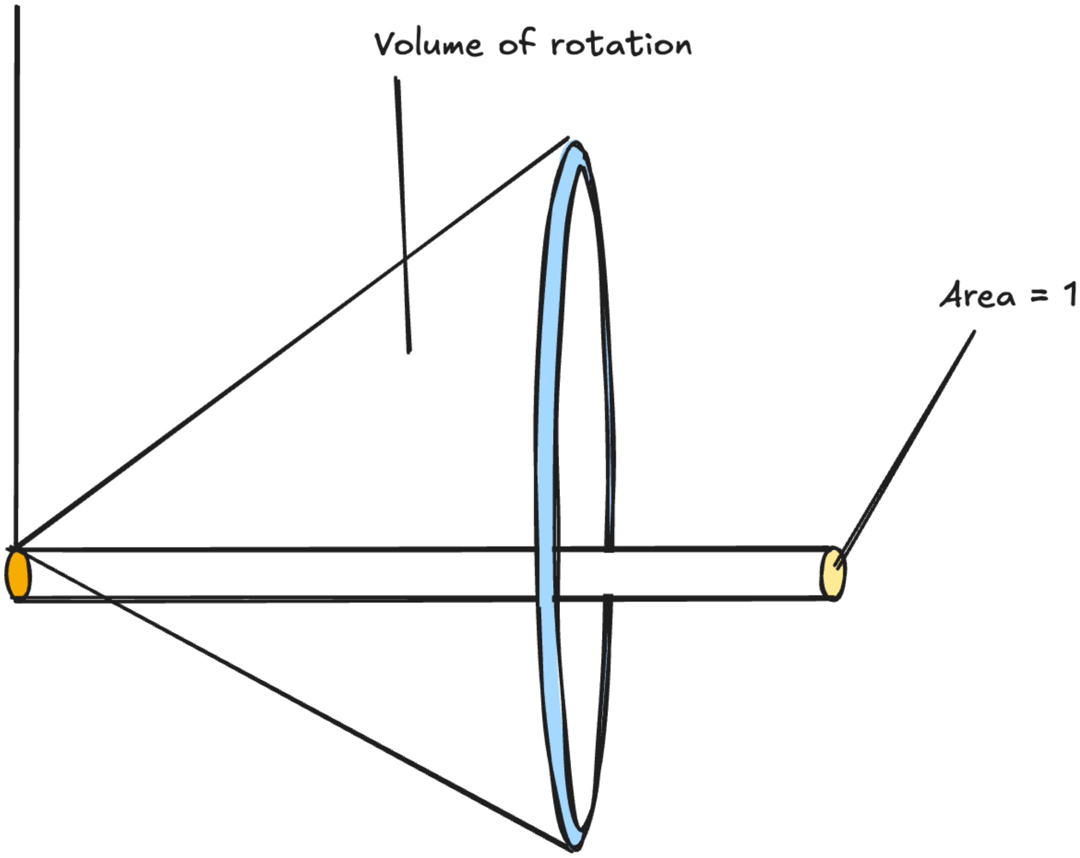


Figure 11: If Newton needs to construct a two-dimensional x-axis in order to rigorously compare how the area changes as a function of x, then Newton would have needed to construct a three-dimensional x-axis when considering a volume of rotation.

2.6. §42

> *To find the Converſe of theſe Things.*
>
> 42. But if upon the contrary, from the Area, or Length, &c. of any Curve being given, the Length of the Baſe AB be required, then you muſt extract the Root *x*, out of the Equations which have been found by the preceding Rules.

If we know the arc length z as a function of x, finding an expression of x as a function of z is equivalent to finding a series expansion of $\sin(\theta)$ in terms of θ.

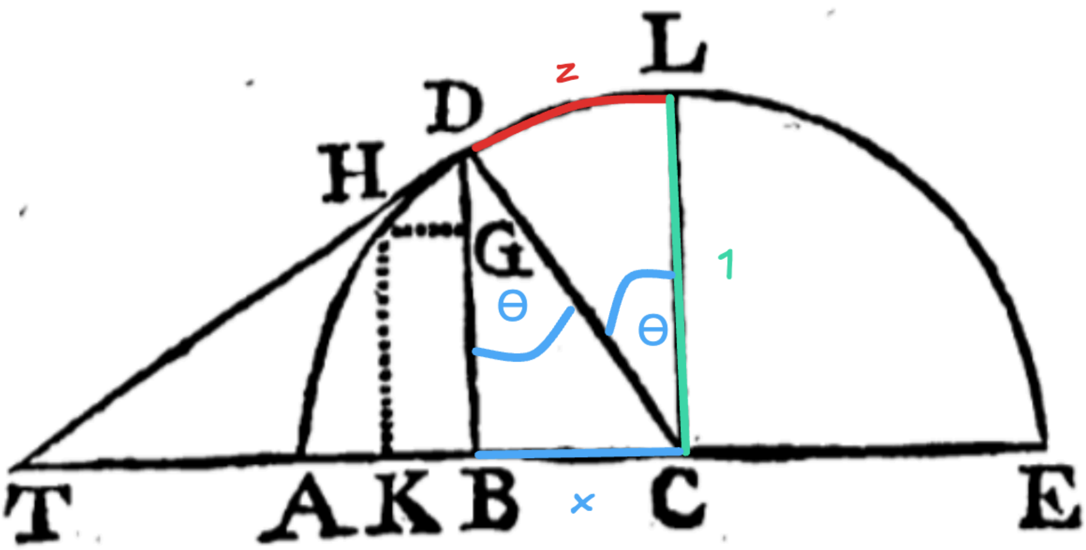


Figure 12: $x = \sin(\theta)$, and $z = \theta$

Because we are on a unit circle (see Figure 12), we can safely assume $z = \theta$. Additionally, by the definition of the sine and arcsine, $x = \sin(\theta)$ and $\theta = \arcsin(x)$.

Therefore, the expansion in Equation 22 is actually the expansion of the arcsine:

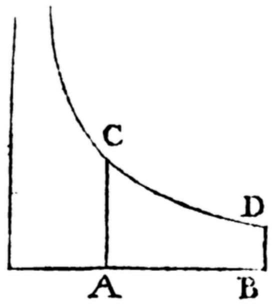
$$z(x) = \theta = \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \tag{23}$$

Inverting this expression will give us the series expansion of the sine:

$$x = \sin(\theta) \tag{24}$$

2.7. §43 How to extract roots numerically.

> *To find the Base from the Area given.*
>
> 43. Thus if from the Area ABDC of the Hyperbola $\left(\frac{1}{1+x} = y\right)$ given I wanted to investigate the Base AB, calling the Area z, I extract the Root of this Equation z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglecting those Terms in which x is of more Dimensions than z is desired in the Quotient.



Newton here introduces a method for inverting series which he will subsequently apply to the series he has just found for sine. In this section, he presents this method using as an example the series for the area under the curve $y = \frac{1}{1+x}$, namely

$$\begin{aligned} z(x) &= \int y(x)dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + ...)dx \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5.. \end{aligned} \tag{25}$$

> As if I would have z to rise to five Dimensions only in the Quotient, I neglect all the Terms $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

So if Newton would have $x(z)$ to '*rise* to the power z^5', that is to say, if he seeks $x(z) = \alpha z + \beta z^2 + \gamma z^3 + \varepsilon z^4 + \delta z^5$, then he neglects all terms in the expansion $z(x)$ higher than x^5 as well. So we neglect all higher terms in Equation 25, put all terms on the LHS:

$$z - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{5}x^5 = 0 \tag{26}$$

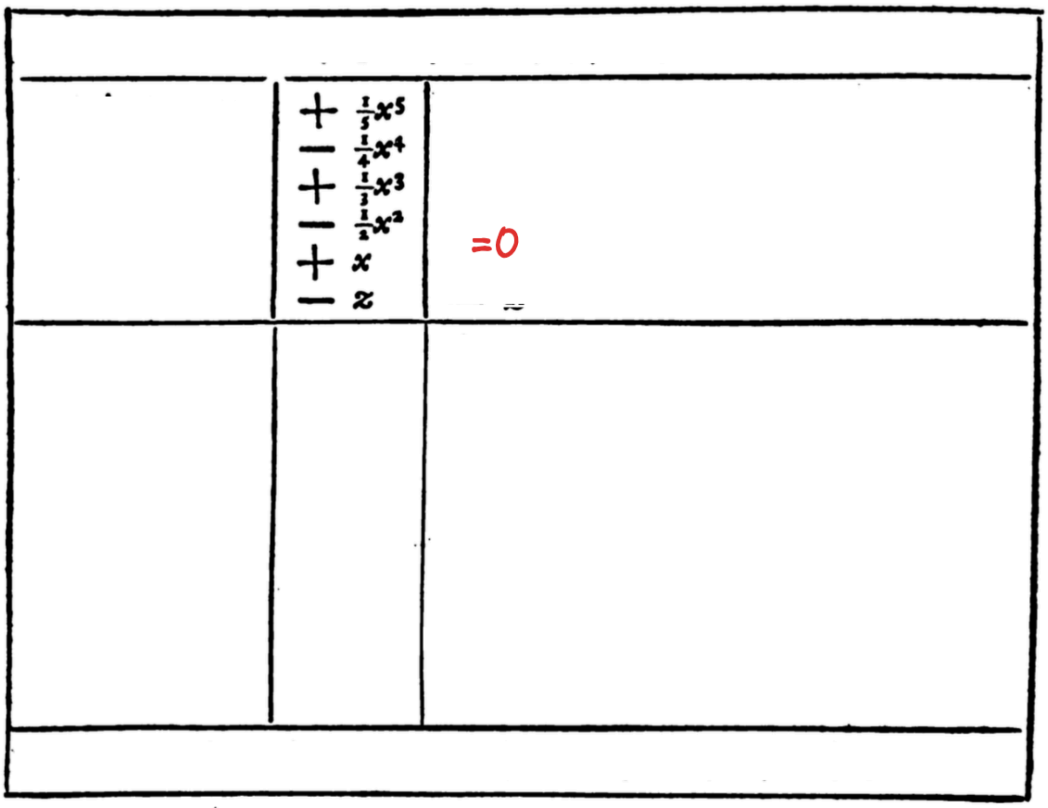


Figure 13: We put the LHS of Equation 26 in the middle of the table.

The first step is to group together all terms in which x has a higher order than z, meaning all terms from $-\frac{1}{2}x^2$ onwards, move them to the other side and call them p. This gives

$$x = z + p \tag{27}$$

where p is some series in terms of z. In order to find out what the next term in this series is, we plug in this substitution for x into Equation 25 to obtain

$$\begin{aligned} z = {} & (z+p) - \frac{1}{2}(z+p)^2 + \frac{1}{3}(z+p)^3 - \frac{1}{4}(z+p)^4 + \dots \\ & = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \dots \\ & +p - zp + z^2p - z^3p + \dots \\ & -\frac{1}{2}p^2 + zp^2 - \dots \\ & \dots \end{aligned} \tag{28}$$

which are the terms displayed in the upper half of Newton's table (see Figure 14). Discarding all rows of terms except the first two gives

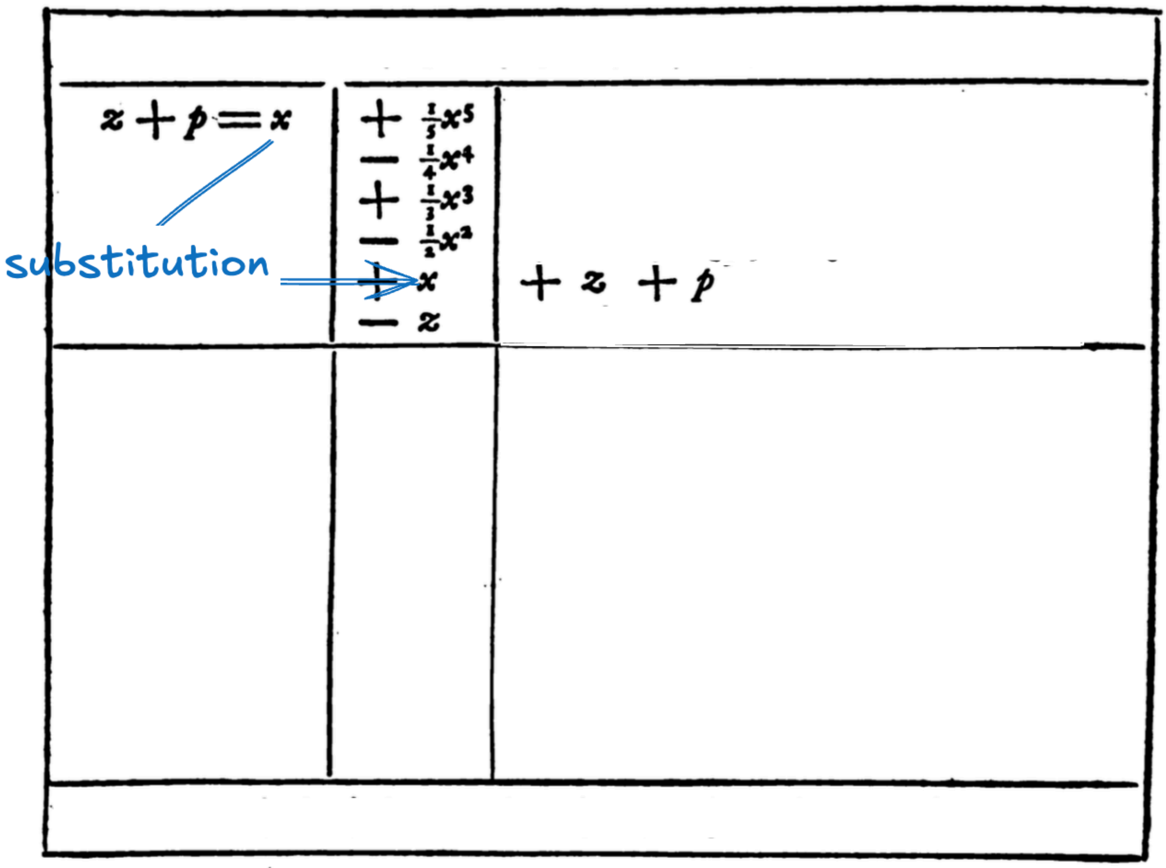


Figure 14: Using substitution $z + p = x$ in one of the terms of the expansion of $z(x)$

$z+p=x$	$+\frac{1}{5}x^5$	$+\frac{1}{5}z^5$ &c.
	$-\frac{1}{4}x^4$	$-\frac{1}{4}z^4 - z^3p$ &c.
	$+\frac{1}{3}x^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$ &c.
	$-\frac{1}{2}x^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+x$	$+z + p$
	$-z$	$-z$

Figure 15: Using substitution $z + p = x$ in all of the terms of the expansion of $z(x)$, and neglecting higher order terms.

$$\begin{aligned} z = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \ldots \\ +p - zp + z^2p - z^3p + \ldots \end{aligned} \tag{29}$$

which can be re-written as

$$\frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}z^4 - \ldots = p - zp + z^2p - z^3p + \ldots \tag{30}$$

This gives us the following expression for p:

$$p = \frac{\frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{4}z^4 - \ldots}{1 - z + z^2 - z^3 + \ldots} \tag{31}$$

We keep only the first term of the series in both the denominator and the numerator. This gives $p = \frac{1}{2}z^2$, but of course, this is not the full expression for p yet, so we need to write:

$$p = \frac{1}{2}z^2 + q \tag{32}$$

Next up, we need to find an expression for q. We do this by following the same protocol as before, that is, plugging this new-found expression for p into Equation 29, obtaining an equation for q in the same way as before and discarding all but the first term in both numerator and denominator again. Repeating this process, one will find that $q = \frac{1}{6}z^3$, $r = \frac{1}{24}z^4$, $s = \frac{1}{120}z^5$. By pattern recognition, we can see that

$$x = \sum_{k=1}^{n} \frac{z^k}{k!} \tag{33}$$

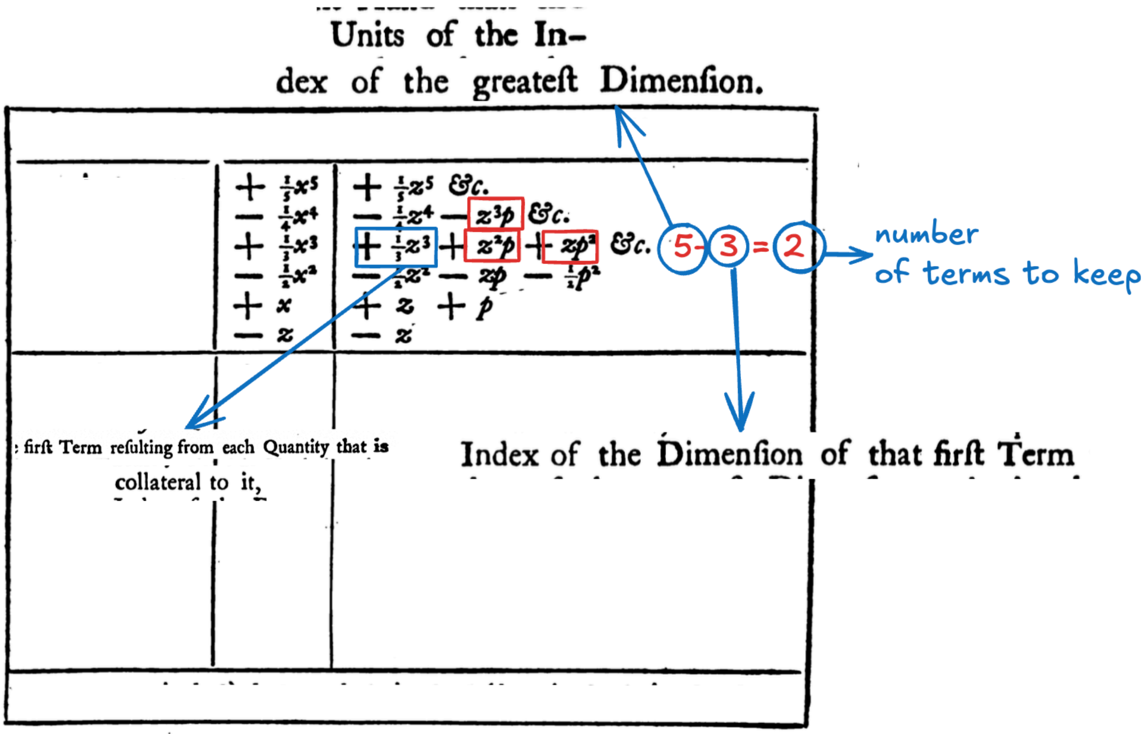


Figure 16: how many units to keep on the RHS

> | | | |
> |---|---|---|
> | $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c. | | |
> | $z + p = x$ | $+\frac{1}{5}x^5$
$-\frac{1}{4}x^4$
$+\frac{1}{3}x^3$
$-\frac{1}{2}x^2$
$+x$
$-z$ | $+\frac{1}{5}z^5$ &c.
$-\frac{1}{4}z^4 - z^3p$ &c.
$+\frac{1}{3}z^3 + z^2p + zp^2$ &c.
$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
$+z + p$
$-z$ |
> | $\frac{1}{2}z^2 + q = p$ | $+zp^2$
$-\frac{1}{2}p^3$
$-z^3p$
$+z^2p$
$-zp$
$+p$
$+\frac{1}{5}z^5$
$-\frac{1}{4}z^4$
$+\frac{1}{3}z^3$
$-\frac{1}{2}z^2$ | $+\frac{1}{4}z^5$ &c.
$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$
$-\frac{1}{2}z^5$ &c.
$+\frac{1}{2}z^4 + z^2q$
$-\frac{1}{2}z^3 - zq$
$+\frac{1}{2}z^2 + q$
$+\frac{1}{5}z^5$
$-\frac{1}{4}z^4$
$+\frac{1}{3}z^3$
$-\frac{1}{2}z^2$ |
> | $1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ | | |

2.8. §44 Justifying ignoring certain terms

> 44. I have laid the Steps of the Reſolution before you, as you ſee, upon the Account of the two following Remarks.
> 1. That in the Subſtitution, I always omit thoſe Terms, which I foreſee will be of no Uſe afterwards. Concerning which this is the
> X x Rule;

Looking at Figure 15, we can see that Newton omits the higher order terms of p of which he sees no use. The rule seems to be that if the middle column has a power of x^n, the terms $z^a p^b$ on the right column have to add up as $a + b = n$.

> Rule; That after the firſt Term reſulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimenſion of that firſt Term wants Units of the Index of the greateſt Dimenſion. As in this Example, where the greateſt Dimenſion is 5, I neglect all the Terms after z^5, I put one after z^4, and two only after z^3. When the Root (*x*) to be extracted, is every where of even or odd Dimenſions, let this be the Rule: That after the firſt Term, reſulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimenſion of that firſt Term, wants Pairs of Units of the Index of the higheſt Dimenſion; or no more than what it wants Ternaries of Units, when the Indexes of the Dimenſions of *x* differ by three Units; and ſo in others.

> 2. When I ſee that *p*, *q*, or *r*, &c. in the laſt reſulting Equation, is found of one Dimenſion only, I ſeek it's Value, that is to ſay the remaining Terms, which are ſtill to be added to the Quotient, by means of Diviſion; as you ſee done here.

> *To find the Baſe from the Length of the Curve given.*

2.8.1. A modern approach to this series inversion

We can take a more modern perspective on Newton's series inversion by starting with the $z(x)$ expression we had before:

$$z(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5.. \tag{34}$$

This time we **assume** that we can write x as a series expansion in terms of z:

$$x(z) = \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 \tag{35}$$

And we search for the coefficients term-by-term by repeatedly substituting Equation 35 into Equation 34. Our single equation turns into 5 equations in 5 variables, one equation for each power of z:

$$
\begin{aligned}
z(x) = &\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right) \\
&-\frac{1}{2}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^2 \\
&+\frac{1}{3}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^3 \\
&-\frac{1}{4}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^4 \\
&+\frac{1}{5}\left(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5\right)^5 \\
&..
\end{aligned} \tag{36}
$$

The powers of z^1 on LHS and RHS must be equal:

$$
\boxed{\alpha = 1} \tag{37}
$$

The powers of z^2 on LHS and RHS must equal:

$$
\begin{aligned}
0 &= \beta + -\frac{1}{2}z^2 \\
&\Rightarrow \boxed{\beta = \tfrac{1}{2}}
\end{aligned} \tag{38}
$$

The same goes for the powers of z^3:

$$
\begin{aligned}
0 &= \gamma + -\frac{1}{2}(2\beta\alpha) + \left(\frac{1}{3}\right)\alpha^3 \\
\gamma &= \frac{1}{2} - \frac{1}{3} \\
&\Rightarrow \boxed{\gamma = \tfrac{1}{6}}
\end{aligned} \tag{39}
$$

The same goes for the powers of z^4:

$$
\begin{aligned}
0 &= \delta + \left(-\frac{1}{2}\right)(\beta^2 + 2\alpha\gamma) + \left(\frac{1}{3}\right)(3\alpha^2\beta) - \left(\frac{1}{4}\right)\alpha^4 \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)^2 + 2(1)\left(\frac{1}{6}\right)\right) + \left(\frac{1}{3}\right)\left(3(1)^2\left(\frac{1}{2}\right)\right) - \left(\frac{1}{4}\right)(1)^4 \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{1}{4} + \frac{1}{3}\right) + \frac{1}{2} - \frac{1}{4} \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{3}{12} + \frac{4}{12}\right) + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta + \left(-\frac{1}{2}\right)\left(\frac{7}{12}\right) + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta - \frac{7}{24} + \frac{12}{24} - \frac{6}{24} \\
0 &= \delta - \frac{1}{24} \\
&\Rightarrow \boxed{\delta = \tfrac{1}{24}}
\end{aligned} \tag{40}
$$

And finally, for the powers of z^5:

The same goes for the powers of z^5:

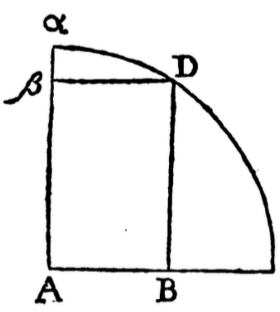
$$
\begin{aligned}
0 &= \varepsilon + \left(-\frac{1}{2}\right)(2\alpha\delta + 2\beta\gamma) + \left(\frac{1}{3}\right)(3\alpha^2\gamma + 3\alpha\beta^2) - \left(\frac{1}{4}\right)(4\alpha^3\beta) + \left(\frac{1}{5}\right)\alpha^5 \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(2(1)\left(\frac{1}{24}\right) + 2\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\right) + \left(\frac{1}{3}\right)\left(3(1)^2\left(\frac{1}{6}\right) + 3(1)\left(\frac{1}{2}\right)^2\right) - \left(\frac{1}{4}\right)\left(4(1)^3\left(\frac{1}{2}\right)\right) + \left(\frac{1}{5}\right)(1)^5 \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{1}{12} + \frac{1}{6}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4}\right)(2) + \frac{1}{5} \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{1}{12} + \frac{2}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{4} + \frac{3}{4}\right) - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon + \left(-\frac{1}{2}\right)\left(\frac{3}{12}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{4}\right) - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon - \frac{1}{8} + \frac{5}{12} - \frac{1}{2} + \frac{1}{5} \\
0 &= \varepsilon - \frac{15}{120} + \frac{50}{120} - \frac{60}{120} + \frac{24}{120} \\
0 &= \varepsilon - \frac{1}{120}
\end{aligned}
\tag{41}
$$

$$\boxed{\varepsilon = \frac{1}{120}}$$

Whence:

$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 + \dots \tag{42}$$

2.9. §45 and §46

> 
>
>
> 45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being ſuppoſed that AB $= x$, αD $= z$, and Aα $= 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c.
>
> 46. And moreover if the Coſine Aβ were required from that Arch given, make Aβ $(= \sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

V: He does not actually give a close form solution, interestingly (i.e involving a $\sum$)

2.10. §47

> *Concerning the Continuation of the Series of the Progressions.*
>
> 47. Let it be observed here, by the bye, that when 5 or 6 Terms of those Roots are known, they may be continued at Pleasure for most Part, by observing the Analogy of the Progression.
>
> Thus you may continue this $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. by dividing the last Term by the following Numbers in Order, 2, 3, 4, 5, 6, 7, &c.
>
> And this $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. by dividing by these Numbers 2×3, 4×5, 6×7, 8×9, 10×11, &c.
>
> And

> And this $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6$, &c. by these 1×2, 3×4, 5×6, 7×8, 9×10, &c.
>
> And this $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. by multiplying by these, *viz.* $\frac{1 \times 1}{2 \times 3}$, $\frac{3 \times 3}{4 \times 5}$, $\frac{5 \times 5}{6 \times 7}$, $\frac{7 \times 7}{8 \times 9}$, &c. And so in others.

A Excerpts from The *Treatise of Quadrature of Curves*

It seems like in [2, §1] Newton is explicitly contrasting his Fluxional approach with the one favoured by Leibniz, Bernoulli, and L'Hôpital (i.e in the *Analyse des infinitement petits*), wherein curves *are* really constisting of many small parts:

Sir *ISAAC NEWTON*'s

TREATISE

OF THE

Quadrature of CURVES.

INTRODUCTION to the Quadrature of Curves.

A. 1 §1

INTRODUCTION to the Quadrature of Curves.

1. I Conſider mathematical Quantities in this Place not as conſiſting of very ſmall Parts; but as deſcrib'd by a continued Motion. Lines are deſcrib'd, and thereby generated not by the Appoſition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies's; Angles by the Rotation of the Sides; Portions of Time by a continual Flux: and ſo in other Quantities. Theſe Geneſes really take Place in the Nature of Things, and are daily ſeen in the Motion of Bodies. And after this Manner the Ancients, by drawing moveable right Lines along immoveable right Lines, taught the Geneſis of Rectangles.

A. 2 §2

2. Therefore conſidering that Quantities, which increaſe in equal Times, and by increaſing are generated, become greater or leſs according to the greater or leſs Velocity with which they increaſe and are generated; I ſought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling theſe Velocities of the Motions or Increments *Fluxions*, and the generated Quantities *Fluents*, I fell by degrees upon the Method of Fluxions, which I have made uſe of here in the Quadrature of Curves, in the Years 1665 and 1666.

3. Fluxions are very nearly as the Augments of the Fluents generated in equal but very ſmall Particles of Time, and, to ſpeak accurately, they are in the *firſt Ratio* of the naſcent Augments; but they may be expounded by any Lines which are proportional to them.

Derivative of x^n from the principle of moments.

Note it requires to let *o* vanish at the end of the proof.

11. *Let the Quantity x flow uniformly, and let it be propoſed to find the Fluxion of x^n.*

In the ſame Time that the Quantity x, by flowing, becomes $x + o$, the Quantity x^n will become $\overline{x + o}|^n$, that is, by the Method of infinite Series's, $x^n + nox^{n-1} + \frac{n^2 - n}{2}oox^{n-2} +$ &c. And the Augments o and $nox^{n-1} + \frac{n^2 - n}{2}oox^{n-2} +$ &c. are to one another as 1 and $nx^{n-1} + \frac{n^2 - n}{2}ox^{n-2} +$ &c.

Now let theſe Augments vaniſh, and their ultimate Ratio will be 1 to nx^{n-1}.

The following is the first appearance of the “Moment” in Newton’s work. Newton introduces the term “Moment” in order to prove that:

$$\begin{aligned} x^3 - xy^2 + a^2 z - b^3 = 0 \\ \Rightarrow 3\dot{x}x^2 - 2xy\dot{y} + -\dot{x}y^2 + a^2\dot{z} = 0 \end{aligned} \tag{43}$$

PROP. I. PROB. I.

15. *An Equation being given involving any Number of flowing Quantities, to find the Fluxions.*

SOLUTION.

Let every Term of the Equation be multiplied by the Index of the Power * of every flowing Quantity that it involves, and in every Multiplication change the Side or Root of the Power into it's Fluxion, and the Aggregate of all the Products with their proper Signs, will be the new Equation.

EXPLICATION.

16. Let a, b, c, d, &c. be determinate and invariable Quantities, and let any Equation be propoſed involving the flowing Quantities z, y, x, &c. as $x^3 - xy^2 + a^2z - b^3 = 0$. Let the Terms be firſt multiplied by the Indexes of the Powers of x, and in every Multiplication for the Root, or x of one Dimenſion write $\dot{x}$, and the Sum of the Factors will be $3\dot{x}x^2 - \dot{x}y^2$. Do the ſame in y, and there ariſes $-2xy\dot{y}$. Do the ſame in z, and there ariſes $aa\dot{z}$. Let the Sum of theſe Products be put equal to nothing, and you'll have the Equation $3\dot{x}x^2 - \dot{x}y^2 - 2xy\dot{y} + aa\dot{z} = 0$. I ſay the Relation of the Fluxions is defin'd by this Equation.

DEMONSTRATION.

17. For let o be a very ſmall Quantity, and let $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ be the Moments, that is the momentaneous ſynchronal Increments of the Quantities z, y, x. And if the flowing Quantities are juſt now z, y, x, then after a Moment of Time, being increas'd by their Increments $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, theſe Quantities ſhall become $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$: which being wrote in the firſt Equation for z, y and x, give this

In §20 he justifies why it is useful to put one of the fluxions equal to Unity. It seems like putting the fluxion of z equal to unity (and discarding the higher fluxions) is equivalent to finding the derivative with respect to z:

Equation $x^3 + 3x^2o\dot{x} + 3xoo\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xo\dot{y}y - 2\dot{x}o^2\dot{y}y - xo^2\dot{y}\dot{y} - \dot{x}o^3\dot{y}\dot{y} + a^2z + a^2o\dot{z} - b^3 = o$.

Subtract the former Equation from the latter, divide the remaining Equation by o, and it will be $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = o$. Let the Quantity o be diminiſhed infinitely, and neglecting the Terms which vaniſh, there will remain $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = o$. Q. E. D.

A. 3 §18-20

A fuller Explication.

18. After the ſame manner if the Equation were $x^3 - xy^2 + aa\sqrt{ax - y^2} - b^3 = o$, thence would be produced $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + aa\overset{\cdot}{\sqrt{ax - y^2}} = o$. Where if you would take away the Fluxion $\overset{\cdot}{\sqrt{ax - y^2}}$, put $\sqrt{ax - y^2} = z$, and it will be $ax - y^2 = z^2$, and by this Propoſition $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, or $\frac{a\dot{x} - 2\dot{y}y}{2z} = \dot{z}$, that is $\frac{a\dot{x} - 2\dot{y}y}{2\sqrt{ax - yy}} = \overset{\cdot}{\sqrt{ax - yy}}$. And thence $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + \frac{a^3\dot{x} - 2a^2\dot{y}y}{2\sqrt{ax - yy}} = o$.

19. And by repeating the Operation, you proceed to ſecond, third and ſubſequent Fluxions. Let $zy^3 - z^4 + a^4 = o$ be an Equation propos'd, and by the firſt Operation it becomes $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = o$; by the ſecond $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}^2z^2 = o$, by the third, $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = o$.

20. But when one proceeds thus to ſecond, third and following Fluxions, it is proper to conſider ſome Quantity as flowing uniformly, and for it's firſt Fluxion to write Unity, for the ſecond and ſubſequent ones, nothing. Let there be given the Equation $zy^3 - z^4 + a^4 = o$, as above; and let z flow uniformly, and let it's Fluxion be Unity: then by the firſt Operation it ſhall be $y^3 + 3z\dot{y}y^2 - 4z^3 = o$; by the ſecond $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 12z^2 = o$; by the third $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z = o$.

But in Equations of this Kind it muſt be conceived that the Fluxions in all the Terms are of the ſame Order, *i. e.* either all of the firſt Order $\dot{y}$, $\dot{z}$; or all of the ſecond $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$; or all of the third $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. And where the Caſe is otherwiſe the Order is to be completed by means of the Fluxions of a Quantity that flows uniformly, which Fluxions are underſtood. Thus the laſt Equation,

tion, by completing the third Order, becomes $9z\dddot{y}y^2 + 18\dot{z}\ddot{y}^2y + 3\hat{z}\dot{y}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 24z\dot{z}^3 = 0$.

B Glossary

These definition are adapted from Robert Pyke [3]

Moment The amount a fluent changes in a small amount of time due to its fluxion. (moment = fluxion × time).

In [2, §17], Newton gives an explicit definition:

> *For let o be a very small Quantity, and let $o\dot{z}, o\dot{y}, o\dot{x}$ be the Moments, that is the momentaneous synchronal Increments of the Quantities z, y, x. And if the flowing Quantities are just now z, y, x, then after a Moment of Time, being increas'd by their Increments $o\dot{z}, o\dot{y}, o\dot{x}$, these Quantities shall become $z + o\dot{z}, y + o\dot{y}, x + o\dot{x}$*

He continues to describe the procedure involving taking $o \to 0$:

Fluxion the velocity (or "celerity") at which the fluent is moving. Denoted $\dot{x}, \dot{y}, \dot{z}$. See [2, §13]

Fluent something that changes ('moves'), e.g. points, lines, planes

Superficies Area

? Conceptual Question

V: While it is not surprising that the different choices of variables in §38 and §37 yield different power series, it does seem a bit strange that they have different powers appearing - i.e it is not just a change in coefficients. How can we explain this? Why does the first expansion have half-odd integer powers but the second one has odd integers.

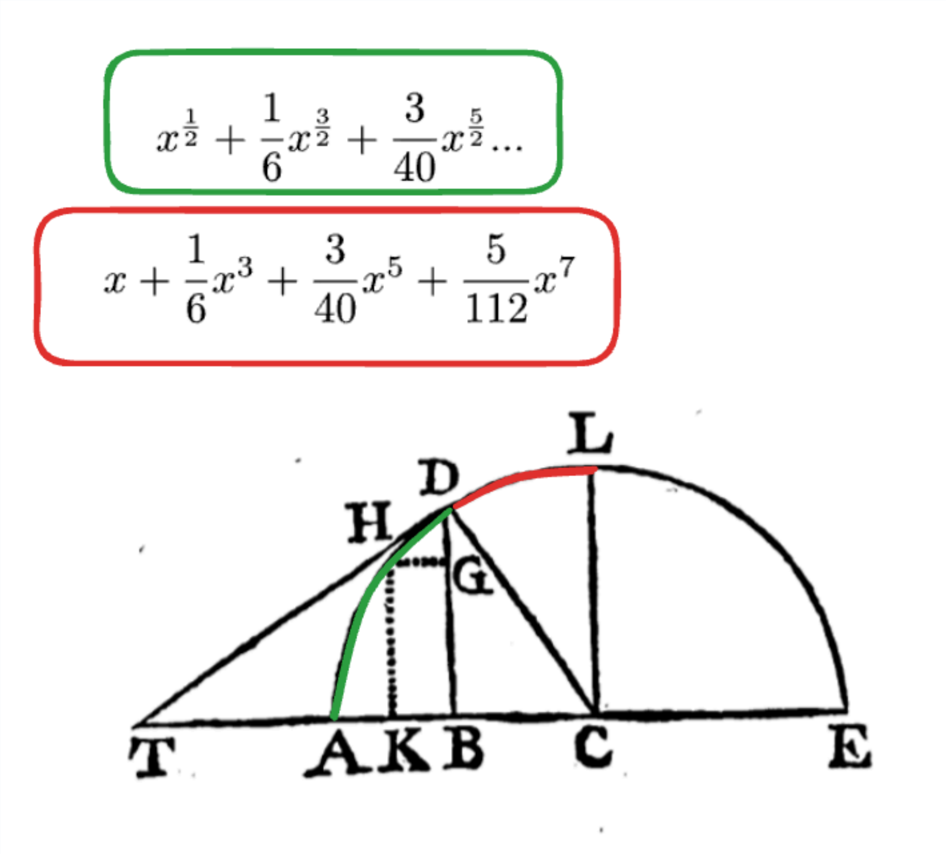


Figure 17:

Since the expansions are just integrals of the moments, it must be due to the different moments involved in §38 and §39.

In §38, the moment $\dot{z}dt$ is $\frac{1}{2\sqrt{x}}(1-x)^{-\frac{1}{2}}dt$

? Conceptual Question

V: In §38, can't we just take $z \to \frac{\pi}{2} - z$ and $x \to \frac{1}{2} - x$ to derive the result. Do we really need to start from scratch?.

Bibliography

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