deadlines
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report: 16th of June

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DRAFT
P3 -group 7

Isaac Newton: Finding the Arcsine

Newton Uses Integration to Find Power Series Expansion for Arcsine and then Inverts it

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Contents

1. Glossary

These definition are taken from Robert Pyke [1]

Moment The amount a fluent changes in a small amount of time due to its fluxion. (moment = fluxion × time).

Fluxion the velocity at which the fluent is moving

Fluent something that changes ('moves'), e.g. points, lines, planes

Superficies Area

2. Introduction

We provide a line-by-line analysis and modern re-interpretation of a section on the infinite series for the sine, taken from an English translation of Newton's *Analysis by means of Equations with an infinite number of terms*, first published in Latin in 1711 [2].

We feel that Newton's diagrams suffer from the use of too many letters, distracting the modern reader from the clarity of his argument. We will use letters sparingly in our interpretation, and rely instead on colours instead.

2.1. How to understand "Moments"

E: I'm a little unsure about Newton's meaning of moment… In one of the modern texts it says that Newton considers the moment of the arc αD to be DH, i.e. dz, and the moment of the base AB the part BK, i.e. dx (these letters referring to Newton's second figure). So that made me think the moment in his view is actually the infinitesimal increment of a quantity. V: Yes, I think we should proceed by treating Moments as infinitesimal line segments rather than derivatives. Essentially because derivatives are what Newton calls Fluxions. V: This interpretation also explains why Newton does not mention Fluxions in this passage. Since Fluxions are always numerically equal (but dimensionally different from) the Moments, it may be confusing to explicitly mention them in the same passage.

For our glossary, we have taken the definition of "Moment" from Robert Pyke [1], but this is not so evident from a first glance at Newton's text. If a Moment is fluxion $\times$ time, as Pyke would have it, where are all the dts (or equivalently - the small quantities *o*) in the objects Newton calls Moments?

For a section of the *de Analysi* in which Newton talks more explicitly about Moments, see Section 8.1 Let's take a few examples. In §37 Newton writes that the area x (Figure 2) is described by the Moment "1". To our modern eyes this "1" might be most readily understood as the derivative w.r.t t of $x = t$. But if this really were the case, why call this a Moment and not a Fluxion? Given that Newton has reserved this special technical term for the Fluxion, it is more reasonable to understand this Moment "1" as "$1dt$", where the little increment of time is implicit.

Another example: in §38 we appear faced with an apparent contradiction. First, the "Moment of the Arch AD" is supposed to be the infinitesimal line segment HD (Figure 2). Next, the *very same moment* is designated $\frac{\sqrt{x-x^2}}{2x-x^2}$, which looks a lot like an expression of the *derivative* of the arch length with respect to the x coordinate, $\frac{dz}{dx}$.

TODO: Add clearer diagram for what dz is here.

What is going on? Do moments have a dual nature, where they are sometimes derivatives and sometimes line segments? Our best guess is that, like the $BK = 1$ example above, Newton leaves out the small increment of *time* implicit in his Moment definitions, so we should really understand his moment as being $\frac{\sqrt{x-x^2}}{2x-x^2}dt$ - which is equal to $\frac{\sqrt{x-x^2}}{2x-x^2}dx$ when $x = t$.

In fact, this last equality, $\frac{\sqrt{x-x^2}}{2x-x^2}dt = \frac{\sqrt{x-x^2}}{2x-x^2}dx$, follows from $x = t$. Newton *always* assumes a "uniform" motion of x, and since in §37 we find BK=1, $x = t$ follows. And it is this last expression that Newton justifies in §40, the final red herring when it comes to understand Moments. At first glance, §40 appears to suggest that Moments are a derivative - because they always *have a dimensionality one less than the form they generate.* If Moments were line (or area, or volume) segments, then surely in §40 Newton would say that the moments would have the *same* dimensionality as the curves they generate. But, like in §37, if we assume that there is always an implicit dt next to the "Unity" that Newton "puts for the Moment", the contradiction disappears. Time t and distance x have the same dimensionality for Newton. Therefore, an implicit dt multiplying every moment will give us the dimensionality allowing us to view moments as infinitesimals, rather than derivatives.

3. Line by line analysis of Newton sine series

> *The Application of what has been ſaid to other Problems of that Kind.*

Newton has just discussed integration and differentiation in the earlier chapters, and is about to show us how these tools will allow us to get a series expansion for the sine.

> 37. Let ABD be any Curve, and AHKB a Rectangle, whoſe Side AH or BK is Unity:

AHKB is a two-dimensional x-axis whose side length is Unity. When considering areas under Curves Newton prefers to consider a 2-dimensional x-axis[1], as opposed to a one-dimensional x-axis.

As we can see from Figure 2, there is an obvious parallel between increasing the area of the rectangle by the 'Moment' 1 and increasing the area of the arbitrary curve by the 'Moment' y.

V: Although this doesn't explain why Newton didn't simply put these two examples on different diagrams - why not have an area $z(x)$ by swept out by the line $y(x)$ dependent on a one-dimensional x axis? Why two-dimensional?

[1] We also saw this in Newton's Treatise of the Quadrature of Curves in presentation P1-8

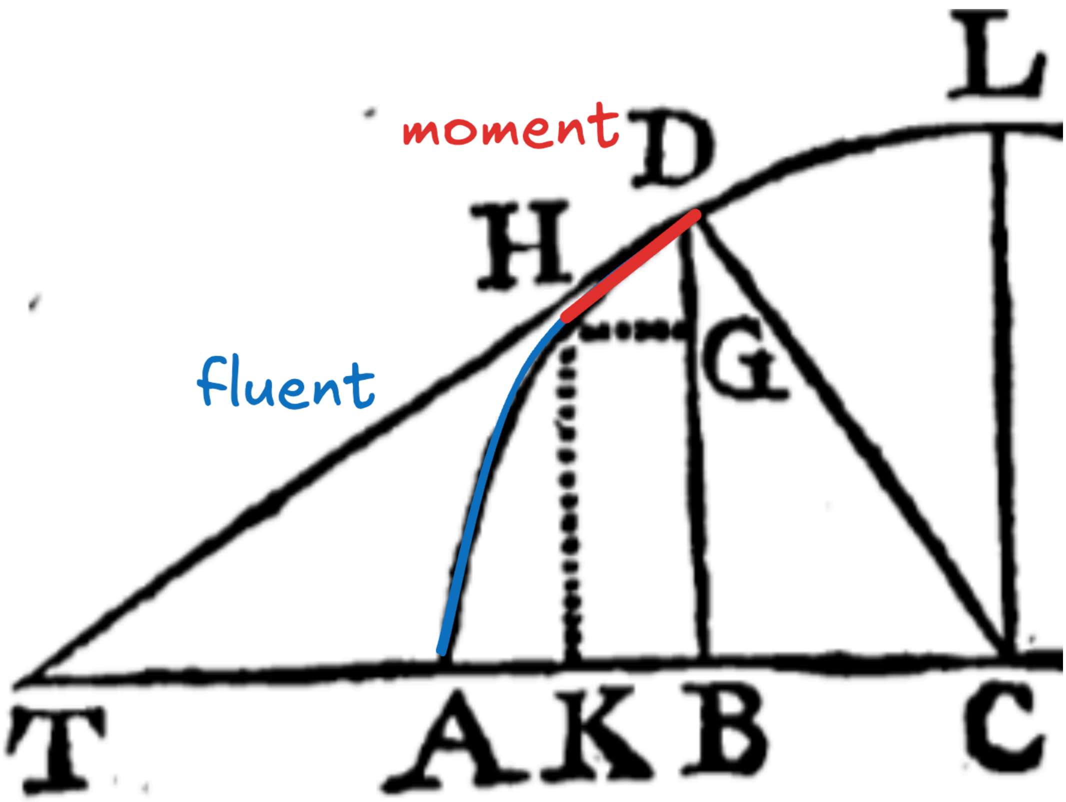
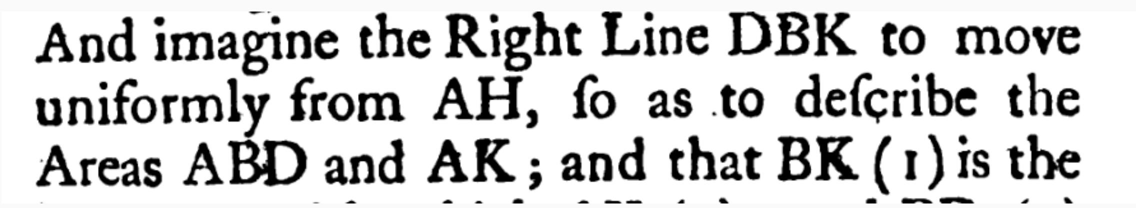


Figure 1: §38 - Newton considers the arc length (blue) to be generated by the moment (red)

> 
> And imagine the Right Line DBK to move uniformly from AH, ſo as to deſcribe the Areas ABD and AK; and that BK (1) is the

Since Newton's calculus fundamentally includes time t implicitly, perhaps it will be illustrative to make the time-dependence explicit. Newton's 'uniform' motion can be satisfied by letting:

$$x = t \tag{1}$$

V: or $x = At + B$ but let's not split hairs at the moment. Although I am pretty sure we need A=1 if Newton is to have his BK = 1

Then, $x = t \Rightarrow \dot{x} = 1$, and thus the Fluxion of x is 1 and the Moment BK is $1dt$. Since this dt would feature in every moment, Newton leaves it out[2]. Newton gives his "1" the dimensionality of a line. This would seem to cast doubt on identifying the moment with the differential area segment, because it is the *derivative* of the area x that has the dimensionality of a line, not the differential area segment dx, but we understand it as follows. If we assume that time and length have the same dimension, then the "1" in the differential area segment $1dt$ will indeed have the dimensionality of a line. So, to conclude, whenever Newton speaks of the "moment" f, we should think of it as an

[2]Not only does he leave out the dt, he gives the "Moment" the dimensionality of what we would understand as the derivative - see §40

infinitesimal segment of the fluent in question, fdt, but with the proviso that Newton is working with units in which time and length have the same dimensionality.

> Areas ABD and AK; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreaſed; and that from the Moment BD

So x is "gradually increased" by $1dt$ (equivalently, $1dx$, or the line of length 1) and y is "gradually increased" by ydt (equivalently, ydx, or the line of length y)

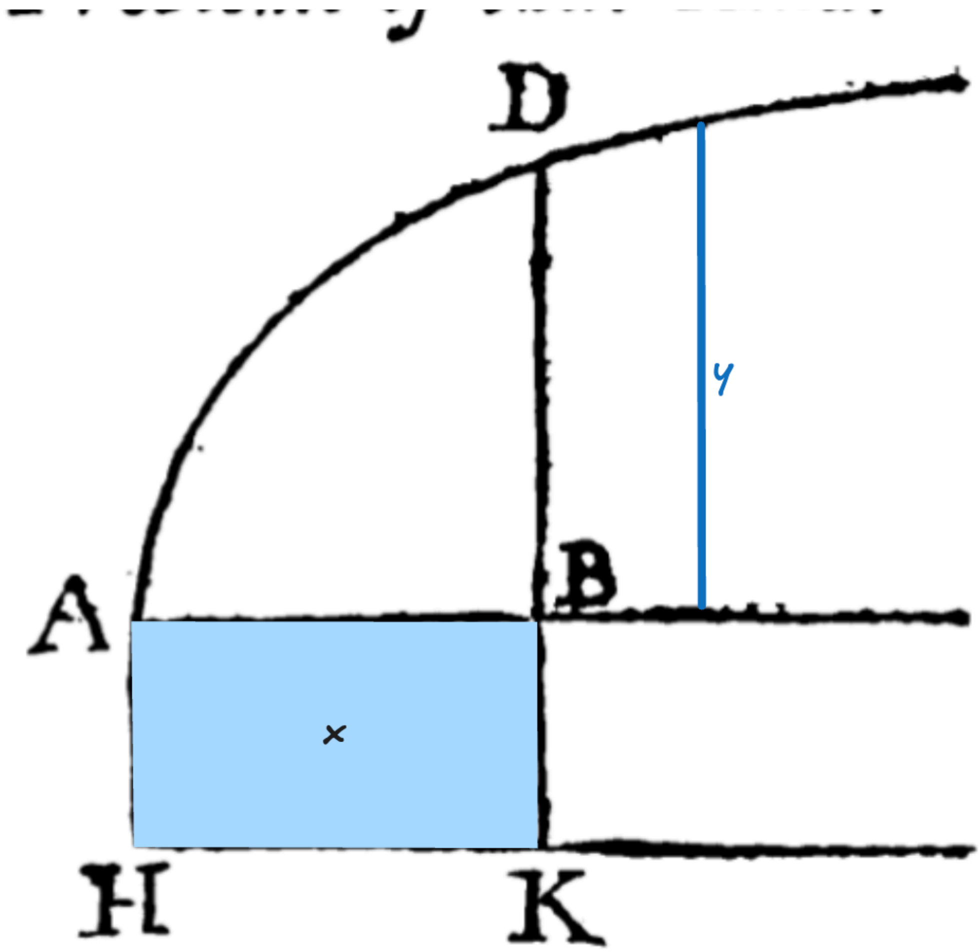


Figure 2: The blue area (x) is "increased continually by the Moment 1". We would understand the "Moment 1" as simply dx but Newton does not use this notation

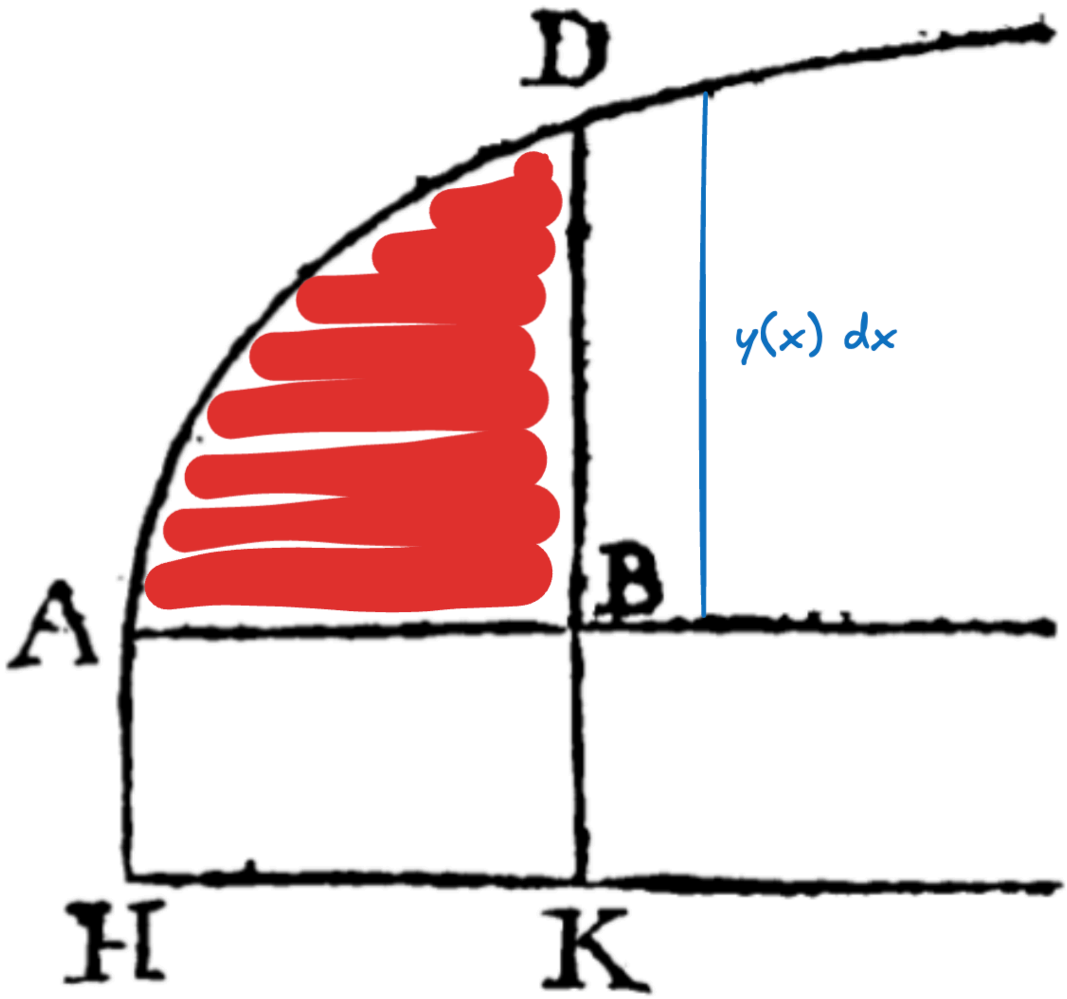


Figure 3: the red area is increased continually by the Moment $y(x)$. Equivalently, it is increased by the infinitesimal area segment $y(x)dx$

> encreafed ; and that from the Moment BD continually given, you can, by Means of the preceding Rules, inveftigate the Area ABD defcribed by it, or compare it with AK (x), which is decribed with the Moment 1.

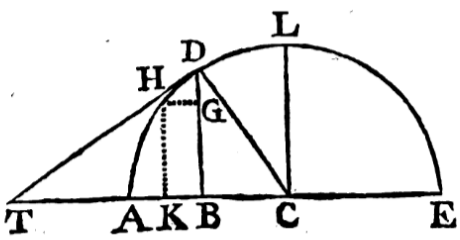
AK is really shorthand for the area $ABKH$. When Newton says "investigate the Area ABD described by it, or *compare* it with $AK(x)$", we read him as drawing attention to how the area under the curve changes *as a function* of x. This may be why Newton wants x to be an Area, for comparing areas makes more intuitive sense than comparing Areas to lines.

> Now by the ſame Means that the Superficies ABD from it's Moment being at all Times given, is diſcovered by the foregoing Rules, by the like Means may any other Quantity be inveſtigated from it's Moment given in like manner. The Thing will be clearer by an Example.

V TODO: reword the following passage for clarity:

In this passage, "Quantity" should be understood as a Fluent, i.e varying with time. If we know the Moment ($y(x)dt$) of a Fluent at all times, we can calculate the Superficies ABD (the integral $\int y(x)dx$). It is striking that Newton still uses an x instead of a t for his independent variable, as it is clear from this discussion that the Moment should be viewed as a function of time. Indeed, since x is supposed to vary uniformly with time, knowing y "at all times" is equivalent to knowing the function $y(x)$. "any Quantity may be investigated from it's Moment" is equivalent to saying - "Any Moment (derivative) can be integrated".

> *To find the Lengths of Curves.*
>
> 38. Let ADLE be a Circle, the Length of whoſe Arch AD is to be inveſtigated. Draw the Tangent DHT, and having compleated the indefinitely ſmall Rectangle HGBK, and put AE = 1 = 2AC, it ſhall be as BK or GH the Moment of the Baſe AB (x) to HD the Moment of the Arch AD :: BT : DT :: BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) :: 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). And ſo



§38 is given as an example of the principle explained in §37, but it uses a lower dimensionality. In §38, line segments are integrated to calculate the total length of arch, whereas in §37 it is area segments that are integrated to create a total area. Nevertheless, Newton considers these examples close enough that one is given as an example of the other.

We find it puzzling that Newton insists on creating a $\frac{1}{2}$-unit circle rather than a unit circle at this point, especially considering that in §39 he considers a unit circle.

When Newton writes "BD ($\sqrt{x - xx}$) : DC $\left(\frac{1}{2}\right)$", the brackets should not be taken to mean multiplication. Instead, the statement "$A(B) : C(D)$" actually means "$\frac{A}{C} = \frac{B}{D}$". E: Yes I suppose that's true although I tend to think more simply about these brackets as indicating equality, like he's saying "A (which by the way is equal to B) stands to C (which by the way is equal to D)..." but of course that has the same implications as what you describe. For example, "$1(BK) : \frac{1}{2\sqrt{x-x^2}}(DH)$" actually means "$\frac{BK}{DH} = 2\sqrt{x - x^2}$", the truth of which will become clear in our following discussion, where we expand, using several illustrations involving two sets of similar triangles and one instance of the Pythagorean theorem, what Newton derives above in a pithy one-liner.

V: What does the "::" mean? Currently I am thinking something like "AND"
E: I think what he's saying there is essentially $\frac{BK}{DH} = \frac{BT}{DT} = \frac{BD}{DC}$, so the "::" means "=".
V: Yes, makes sense!

We start by noticing that the red and green triangles are similar (see Figure 4) since the two triangles share the angle $\angle BDT$ (or $\angle GDH$) and both triangles contain a right angle ($\angle TBD$ and $\angle HGD$), whence:

$$\frac{DT}{BT} = \frac{DH}{GH} \tag{2}$$

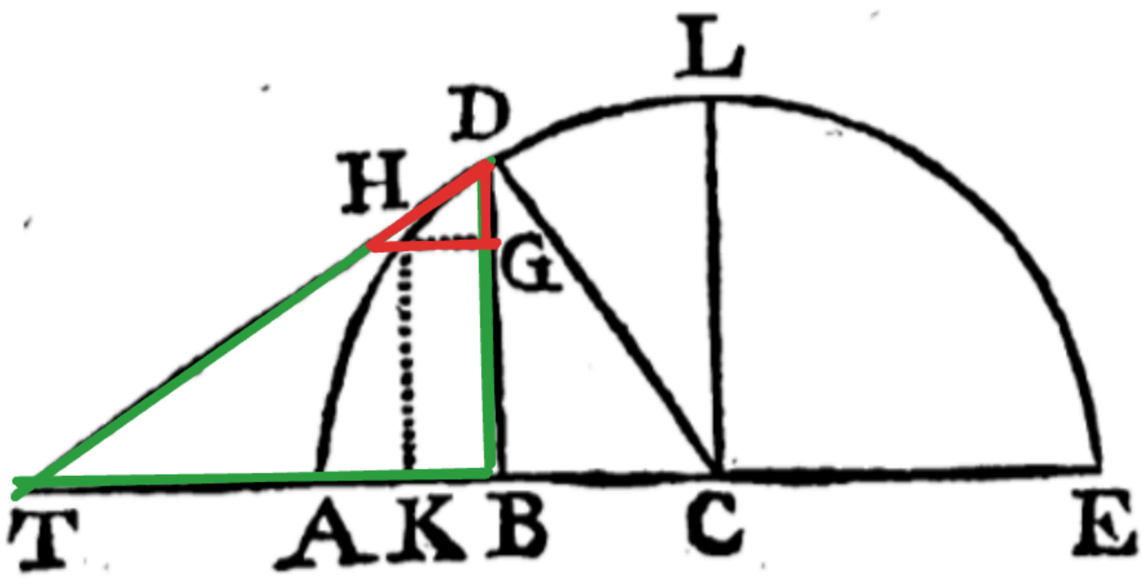


Figure 4: red and green triangles are similar

Next, we notice that the red and green triangles in Figure 5 are also similar, since both contain a right angle ($\angle TBD$ and $\angle DBC$) and $\angle CDB = \angle BTD$. The latter follows from the fact that in $\triangle DBT$, we can see that $90° - \angle BDT = \angle BTD$, and since $\angle CDT = 90°$, we know that $\angle CDB = 90° - \angle BDT$, which we established was equal to $\angle BTD$. TODO: reword for clarity

Therefore:

$$\frac{DT}{BT} = \frac{DC}{BD} \tag{3}$$

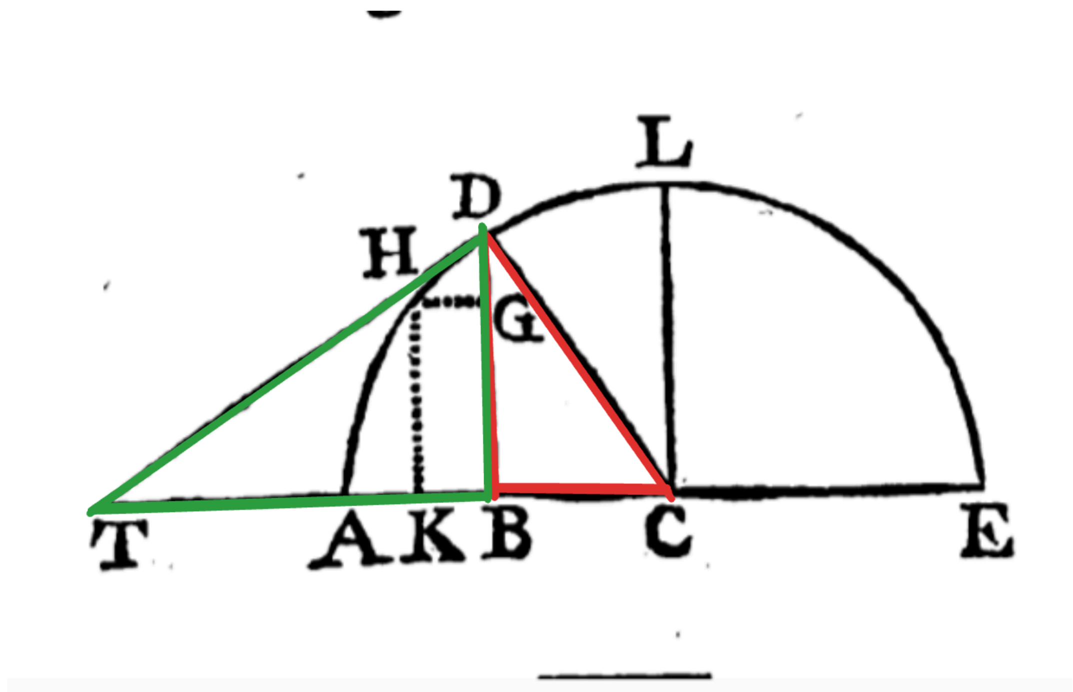


Figure 5: red and green triangles are similar

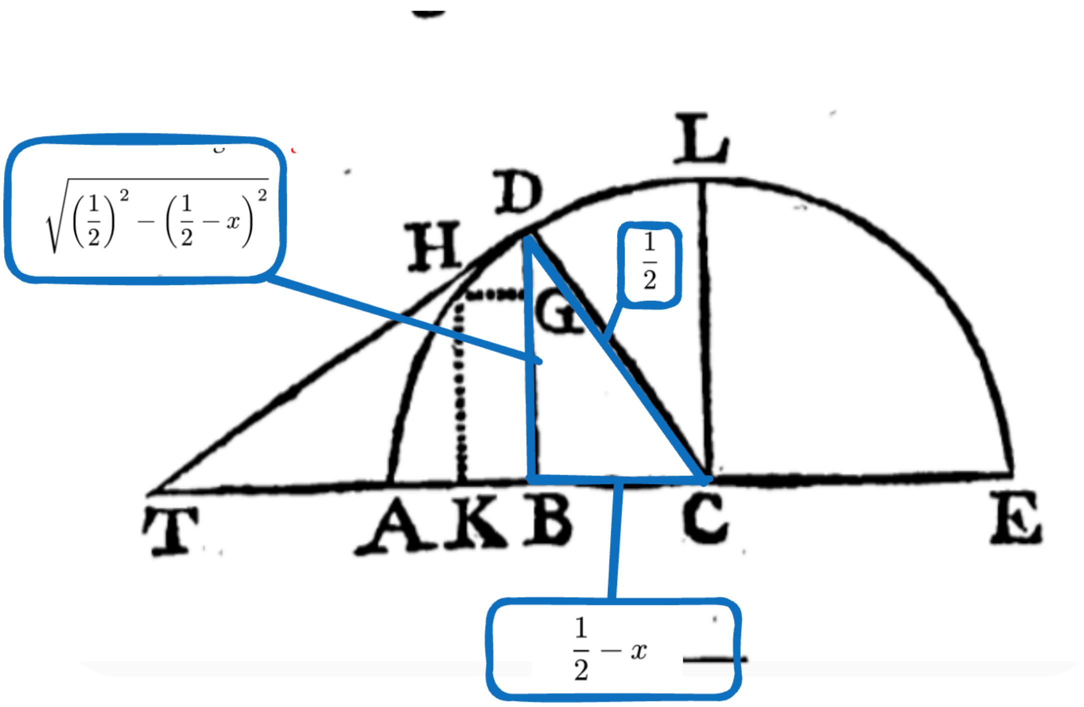


Figure 6: The Pythagorean theorem is used to find the value of the line BD

Next, we use the pythagorean theorem on the blue triangle in Figure 6, to find that:

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} = \sqrt{x - x^2} \tag{4}$$

Finally, by constructing the circle to have a radius of $\frac{1}{2}$, we know that:

$$DC = \frac{1}{2} \tag{5}$$

Combining Equation 2 and Equation 3 to eliminate $\frac{DT}{BT}$ gives:

$$\frac{DH}{GH} = \frac{DC}{BD} \tag{6}$$

Using Equation 4 and Equation 5 to substitute for DC and BD gives:

$$\frac{DH}{GH} = \frac{1}{2\sqrt{x - x^2}} \tag{7}$$

Now we can rewrite our infinitesimal triangle in a way that will be more recognizable to modern readers, expressing the variable length of the arc AD as $z(x)$: TODO: illustrate d x, d a on a diagram

$$\frac{DH}{GH} = \frac{dz}{dx} \tag{8}$$

Therefore:

$$\begin{aligned} \frac{dz}{dx} &= \frac{1}{2\sqrt{x - x^2}} \\ z(x) &= \int \left(\frac{1}{2\sqrt{x - x^2}} \right) dx \end{aligned} \tag{9}$$

In Newton's words, $\frac{1}{2\sqrt{x-x^2}}$ is the 'Moment' - i.e *derivative* - of the Arch AD (or $z(x)$ for us).

We can perform the integral by first writing out the series expansion for $\frac{1}{2\sqrt{x-x^2}}$, then integrating term-by-term.

Next we expand $\frac{1}{2\sqrt{x-x^2}}$ using the Binomial theorem as Newton would have used it:

$$(P + PQ)^{\frac{m}{n}} = A + B + C + D + E + ... \tag{10}$$

where:

$$\begin{aligned} A &= P^{\frac{m}{n}} \\ B &= \frac{m}{n} AQ \\ C &= \frac{m - n}{2n} BQ \\ D &= \frac{m - 2n}{3n} CQ \end{aligned} \tag{11}$$

$$\frac{1}{2\sqrt{x - x^2}} = \frac{1}{2}(x - x^2)^{-\frac{1}{2}} = \frac{x^{-\frac{1}{2}}}{2}(1 - x)^{-\frac{1}{2}} \tag{12}$$

Where $P = 1$ and $Q = -x$, $m = -1$ and $n = 2$

$$\begin{aligned} A &= 1 \\ B &= \frac{1}{2}x \\ C &= -\frac{3}{4}\left(\frac{1}{2}x\right)x = -\frac{3}{8}x^2 \\ D &= \frac{-1-2(2)}{3\times 2}\left(\frac{3}{8}x^2\right)x = \frac{5}{16}x^3 \ldots \end{aligned} \tag{13}$$

So:

$$\begin{aligned} \frac{1}{2\sqrt{x-x^2}} &= \frac{x^{-\frac{1}{2}}}{2}\left(1+\frac{1}{2}x-\frac{3}{8}x^2+\frac{5}{16}x^3\right) \\ &= \frac{x^{-\frac{1}{2}}}{2}+\frac{1}{4}x^{\frac{1}{2}}+\frac{3}{16}x^{\frac{3}{2}}+\frac{5}{32}x^{\frac{5}{2}}+\ldots \end{aligned} \tag{14}$$

Now that we have obtained an expression for the moment of arc $\frac{dz}{dx}$ as a function of x (see Figure 7), we can integrate it term by term to get the arc length

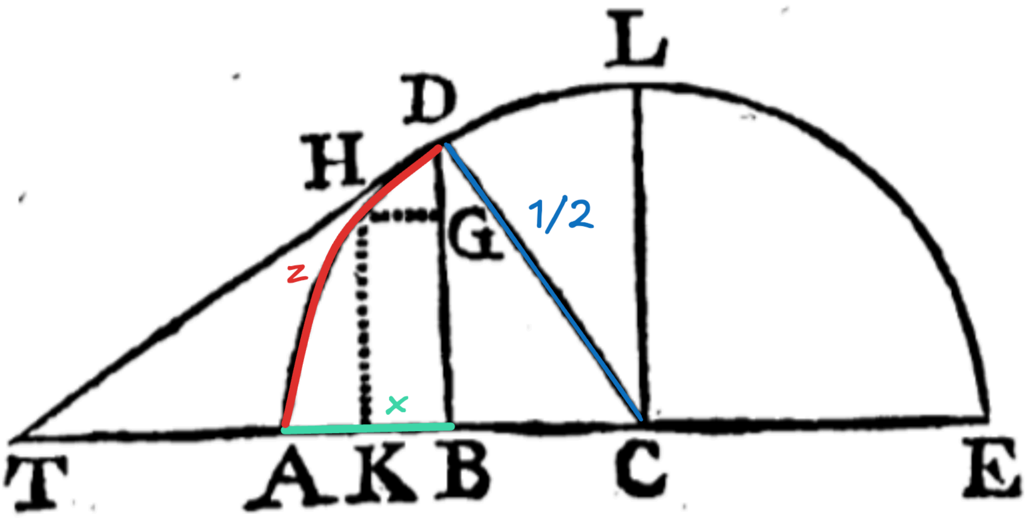


Figure 7: Finding $\frac{dz}{dx}$

$$\frac{dz}{dx} = \frac{1}{2\sqrt{x-x^2}} = \frac{x^{-\frac{1}{2}}}{2}+\frac{1}{4}x^{\frac{1}{2}}+\frac{3}{16}x^{\frac{3}{2}}+\frac{5}{32}x^{\frac{5}{2}}+\ldots \tag{15}$$

$$z(x) = \int z(x)dx = x^{\frac{1}{2}}+\frac{1}{6}x^{\frac{3}{2}}+\ldots \tag{16}$$

4. §39 Change the problem

> 39. After the fame Manner by fuppofing CB to be x, the Radius CA to be 1, you will find the Arch LD to be $x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c.

Now, instead of $CA = DC = \frac{1}{2}$, we have:

$$DC = 1 \tag{17}$$

and instead of defining $AB = x$ we have $BC = x$, so this changes the geometry of the problem quite significantly (see Figure 9). By Pythagoras' theorem we obtain

$$BD = \sqrt{1 - x^2}. \tag{18}$$

Combining Equation 6 and Equation 17 we find that

$$\frac{DH}{GH} = \frac{1}{\sqrt{1 - x^2}} \tag{19}$$

which gives

$$z(x) = \int \left(\frac{1}{\sqrt{1 - x^2}} \right) dx. \tag{20}$$

V: Can use Newton's own theorem here as. well.

As before, applying a series expansion to the integrand allows us to re-write this expression as

$$z(x) = \int \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \ldots \right) dx \tag{21}$$

and integrating term by term gives

$$z(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots \tag{22}$$

5. §40 Dimensionality of Unity

> 40. But it is to be remarked that that Unity which is put for the Moment, is a Superficies, when the Queſtion is about Solids; and a Line when about Superficies; and a Point when it is about Lines (as in this Example.) Neither am I afraid to ſpeak of Unity in Points, or Lines infinitely ſmall, ſince Geometers are wont now to conſider Proportions even in ſuch a Caſe, when they make uſe of the Methods of Indiviſibles.

6. §41: Solids and Centers of Gravity.

> 41. From theſe Things one may gueſs how one ought to proceed in inveſtigating the Superficies and Contents of Solids; and likewiſe the Centers of Gravity.

V: Can we say anything about centers of gravity?

V: If a 2D x axis is needed to allow us to compare the area $\int y(x)dx$ with x in §37, would we need a cylindrical x-axis with area 1 to allow us to compare the volume of rotation of a curve against x

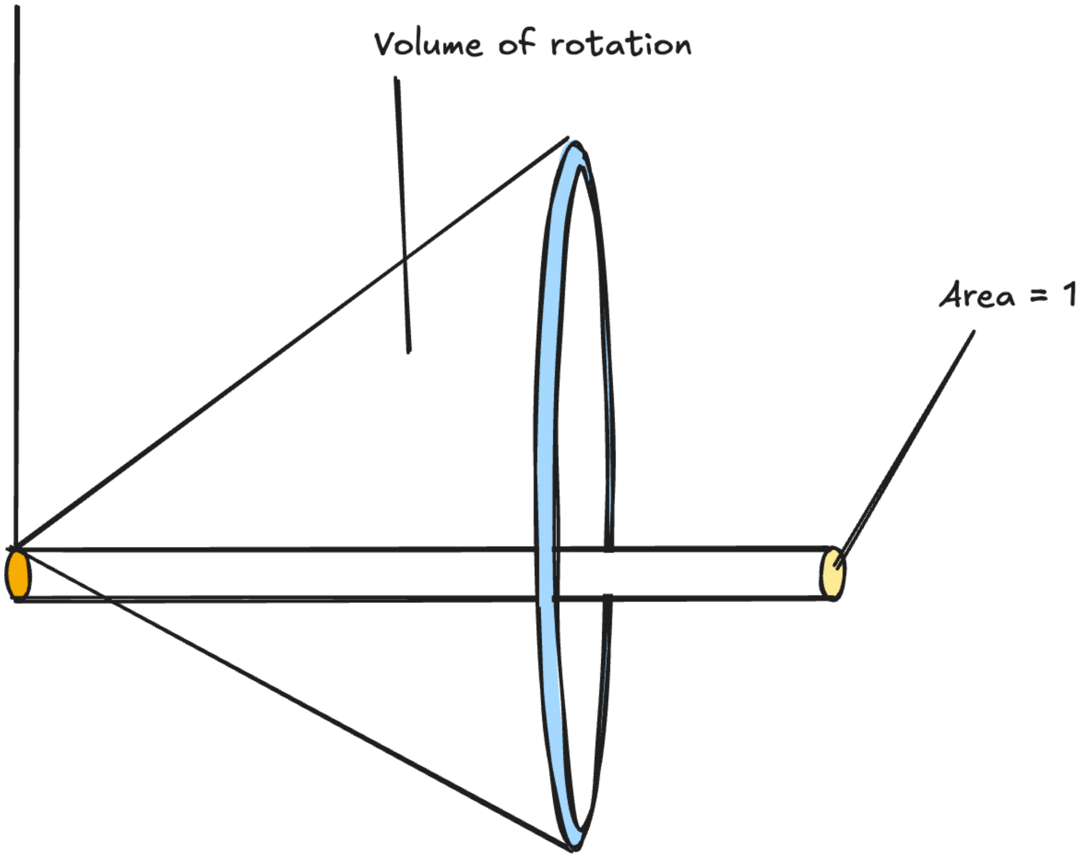


Figure 8: If Newton needs to construct a two-dimensional x-axis in order to rigorously compare how the area changes as a function of x, then Newton would have needed to construct a three-dimensional x-axis when considering a volume of rotation.

> *To find the Converſe of theſe Things.*
>
> 42. But if upon the contrary, from the Area, or Length, *&c.* of any Curve being given, the Length of the Baſe AB be required, then you muſt extract the Root *x*, out of the Equations which have been found by the preceding Rules.

If we know the arc length z as a function of x, finding an expression of x as a function of z is equivalent to finding a series expansion of $\sin(\theta)$ in terms of θ.

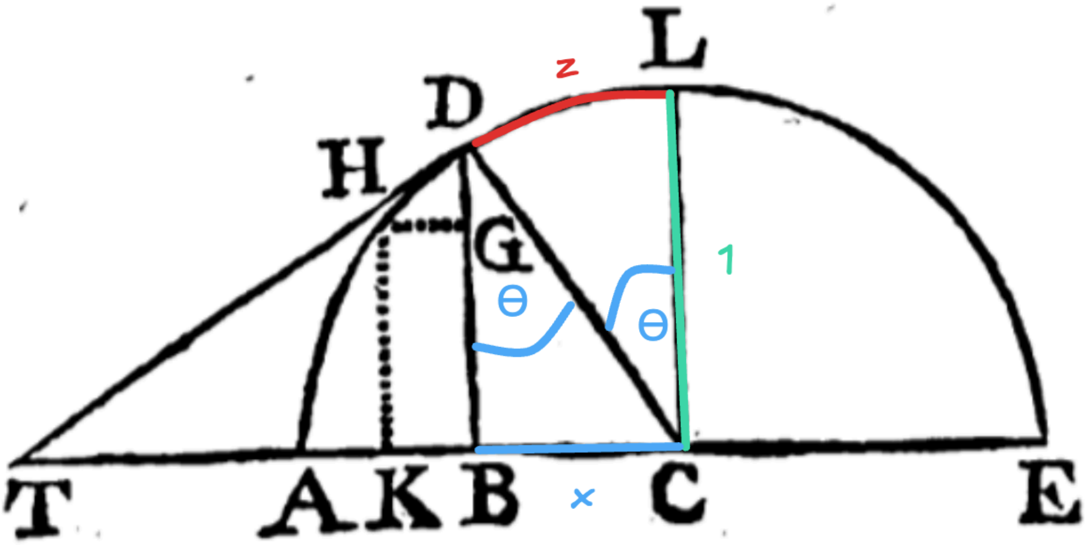


Figure 9: $x = \sin(\theta)$, and $z = 2\pi\theta$

Because we are on a unit circle (see Figure 9), we can safely assume $z = \theta$. Additionally, by the definition of the sine and arcsine, $x = \sin(\theta)$ and $\theta = \arcsin(x)$.

Therefore, the expansion in Equation 22 is actually the expansion of the arcsine:

$$z(x) = \theta = \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots \quad (23)$$

Inverting this expression will give us the series expansion of the sine:

$$x = \sin(\theta) \quad (24)$$

7. §43: How to extract roots numerically.

TODO:

- §43,
- §44,
- §45,
- §46,
- §47,

Bibliography

[1] R. Pyke, 'Fluents and Fluxions'. [Online]. Available: https://www.sfu.ca/~rpyke/fluxions.pdf

[2] Newton, *The Application of what has been said to other Problems of the Kind*. 1711. [Online]. Available: https://books.google.it/books?id=noQ_AAAAcAAJ

8. Appendix

8.1. Analysi, §17

We attach an instructive snippet from the *De Analysi* in which Moments are more explicitly defined: [2, §17]

> *For let o be a very small Quantity, and let* $o\dot{z}, o\dot{y}, o\dot{x}$ *be the Moments, that is the momentaneous synchronal Increments of the Quantities* z, y, x*. And if the flowing Quantities are just now* z*,* y*,* x*, then after a Moment of Time, being increas'd by their Increments* $o\dot{z}, o\dot{y}, o\dot{x}$*, these Quantities shall become* $z + o\dot{z}, y + o\dot{y}, x + o\dot{x}$ *: which being wrote in the first Equation for* z, y *and* x*, give this equation* $x^3 + 3x^2o\dot{x} + 3xo0\dot{x}\dot{x} + 0^3\dot{x}^3 - xy^2 - 0\dot{x}y^3 - 2xo\dot{y}y - 2x^2y^2y - x0^2\dot{y}\dot{y} - \dot{x}0^3\dot{y}\dot{y} + a^2z + a^2oz - b^3 = 0$*. Subtract the former Equation from the latter, divide the remaining Equation by* o *, and it will be* $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = 0$*. Let the Quantity* o *be diminished infinitely, and neglecting the Terms which vanish, there will remain* $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$*. Q. E. D.*

8.2. Delete me

V: I used these in the diagrams above so they might still be useful when making amendments to the diagrams, but we should delete before handing in

$$\frac{1}{2} \qquad \frac{1}{2} - x \qquad \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} \tag{25}$$

$$1 \qquad 1 - x \qquad \sqrt{1 - (1 - x)^2} \tag{26}$$