deadlines
presentation : 9th of June
report: 16th of June

Red text is highly tentative. It may represent an approach that should be substantially revised or deleted

Green text is for TODOs

Question: Text is for as-yet-unresolved questions.

Blue Text is for comments

To modify Victor's illustrations you can here (download and open in excalidraw)

DRAFT
P3 -group 7

Isaac Newton's Infinite Series for the Sine

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Contents

Glossary

Moment The amount a fluent changes in a small amount of time due to its fluxion; moment = fluxion × time. This definition, given by Robert Pyke [1], is not so evident when looking at Newton's text. For example, when Newton says that the area AK in § 37 is described by the Moment "1", we will understand this as the Moment "1 dt", where the little increment of time is always implicit. If we didn't include the dt then the "1" would be a derivative (fluxion), not an infinitesimal area segment (moment). This means that at first glance a lot of the "Moments" that Newton talks about actually look to us like derivatives (i.e the derivative of $x = t$ with respect to t also 1). When Newton writes, in § 38, that $\frac{\sqrt{x-x^2}}{2x-x^2}$ is the *moment* of the Arch AD, he really means $\frac{\sqrt{x-x^2}}{2x-x^2}dx$, which is equal to $\frac{\sqrt{x-x^2}}{2x-x^2}dt$ when $x = t$ ("uniform" motion of x).

E: I'm a little unsure about Newton's meaning of moment… In one of the modern texts it says that Newton considers the moment of the arc αD to be DH, i.e. dz, and the moment of the base AB the part BK, i.e. dx (these letters referring to Newton's second figure). So that made me think the moment in his view is actually the infinitesimal increment of a quantity. V: Yes, I think we should proceed by treating Moments as infinitesimal line segments rather than derivatives. Essentially because derivatives are what Newton calls Fluxions.

Fluxion the velocity at which the fluent is moving [1]

Fluent something that changes (‘moves’), e.g. points, lines, planes [1]

Superficies Area

Introduction

We provide a line-by-line analysis and modern re-interpretation of a section on the infinite series for the sine, taken from an English translation of Newton’s *Analysis by means of Equations with an infinite number of terms*, first published in Latin in 1711 [2].

We feel that Newton’s diagrams suffer from the use of too many letters, distracting the modern reader from the clarity of his argument. We will use letters sparingly in our interpretation, and rely instead on shapes and colours instead.

Line by line analysis of Newton sine series

> *The Application of what has been ſaid to other Problems of that Kind.*

Newton has just discussed integration and differentiation in the earlier chapters, and is about to show us how these tools will allow us to get a series expansion for the sine.

> 37. Let ABD be any Curve, and AHKB a Rectangle, whoſe Side AH or BK is Unity :

AHKB is a two-dimensional x-axis whose side length is Unity. When considering areas under Curves Newton prefers to consider a 2-dimensional x-axis[1], as opposed to a one-dimensional x-axis. This is quite puzzling, considering that in § 38 (see Figure 1) Newton has no trouble imagining that the arc length (blue) is incremented by an infinitesimal moment DH (red).

[1] We also saw this in Newton’s Treatise of the Quadrature of Curves in presentation P1-8

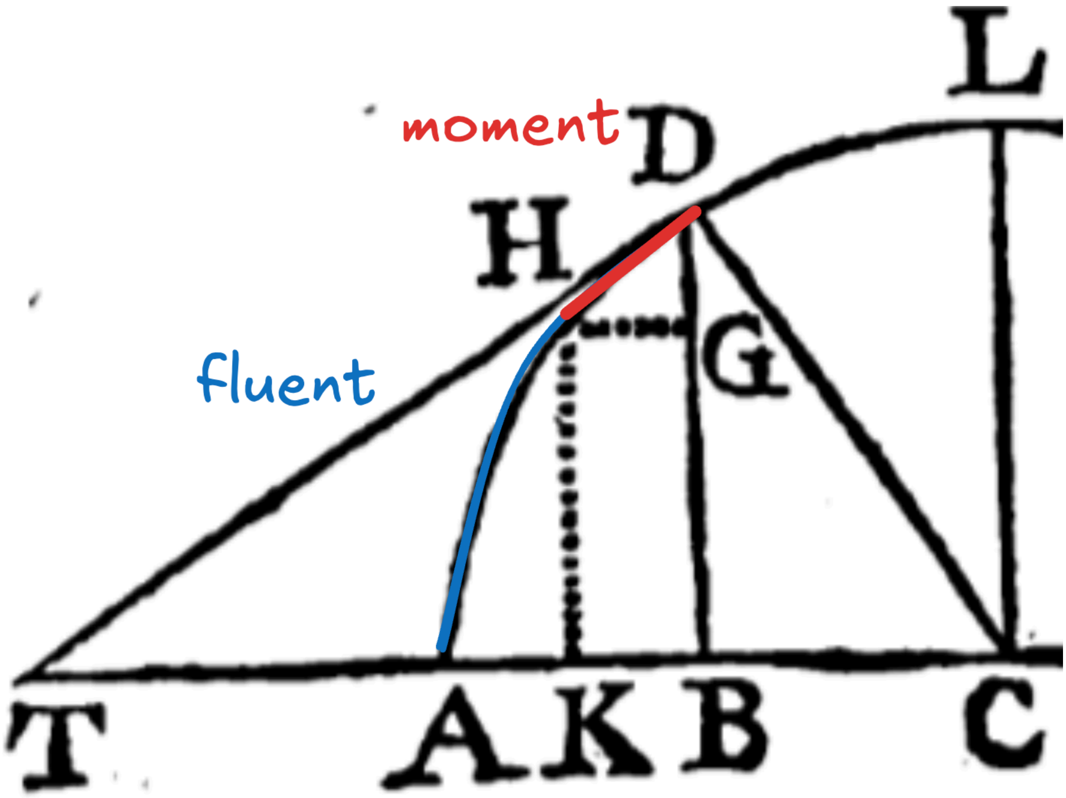
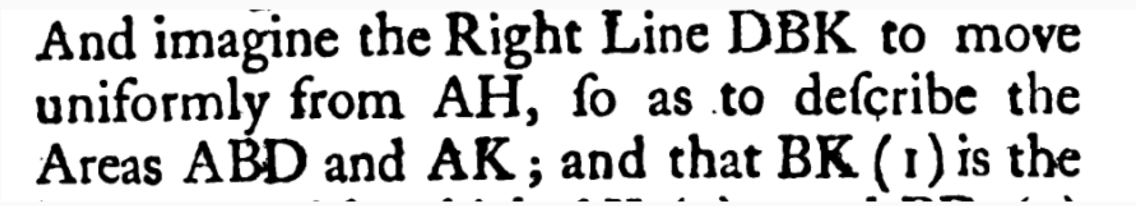


Figure 1: § 38 - Newton considers the arc length (blue) to be generated by the moment (red)

> 
> And imagine the Right Line DBK to move uniformly from AH, ſo as to deſçribe the Areas ABD and AK; and that BK (1) is the

We have to mention that Newton's calculus fundamentally includes time t implicitly. We will try to understand it by making the time-dependence explicit. Newton's 'uniformly' can be satisfied by the equation:

$$x = t \tag{1}$$

V: or $x = At + B$ but let's not split hairs at the moment

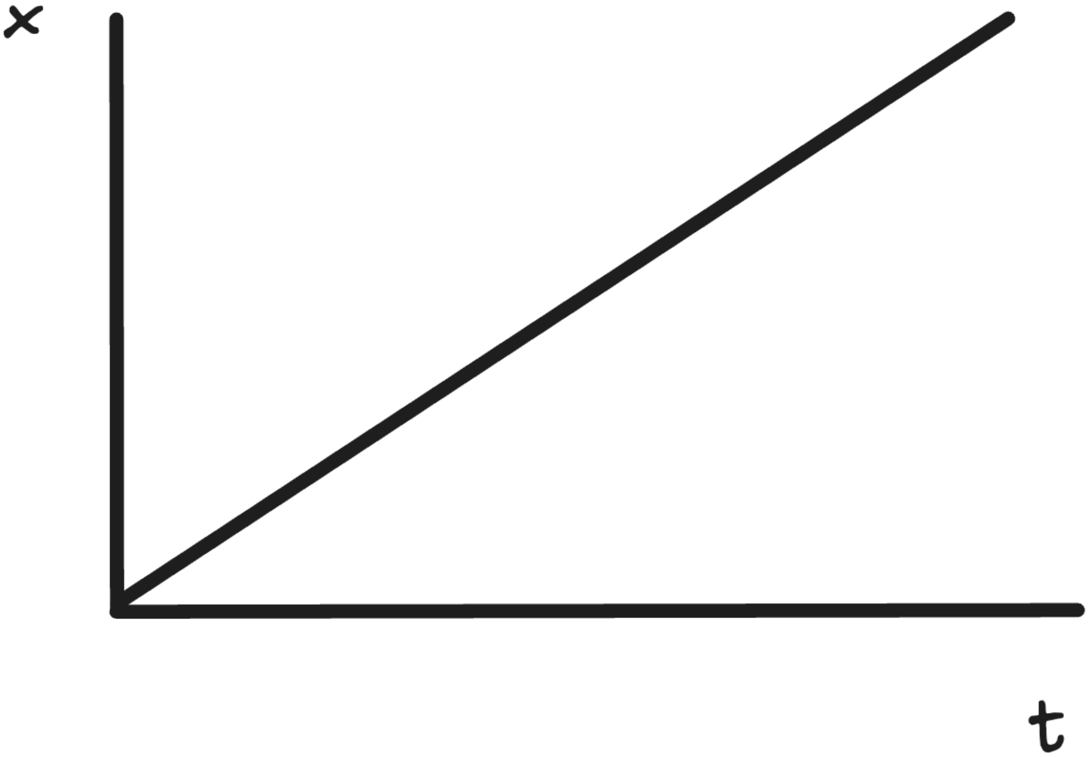
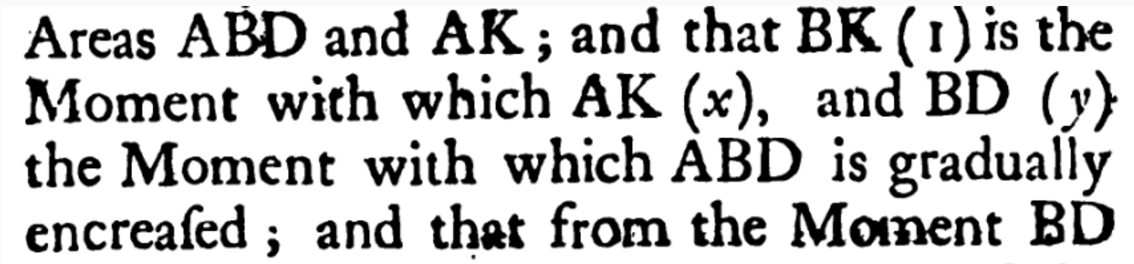


Figure 2: Explicit time-dependence of x, inferred from “imagine the Right Line DBK to move uniformly from AH so as to describe the Areas…”

> 
> Areas ABD and AK; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreaſed; and that from the Moment BD

It appears that by “Moment” Newton is referring to a value f such that the respective areas increase by $f dx$. This is how we understand the *derivative*. Thus, 1 is the Moment by which the area x is continually increased (in blue, Figure 3. And y is the Moment by which the red area (Figure 4) is continually increased.

It is clear from our diagram that x is supposed to have the dimensionality of area, whereas y is supposed to have the dimensionality of length. However, we don’t believe that it is merely an idiosyncrasy that Newton wants his x-axes to be two dimensional. As we can see from our diagram, there is an obvious parallel between increasing the area of the rectangle by the ‘Moment’ 1 and increasing the area of the arbitrary curve by the ‘Moment’ y.

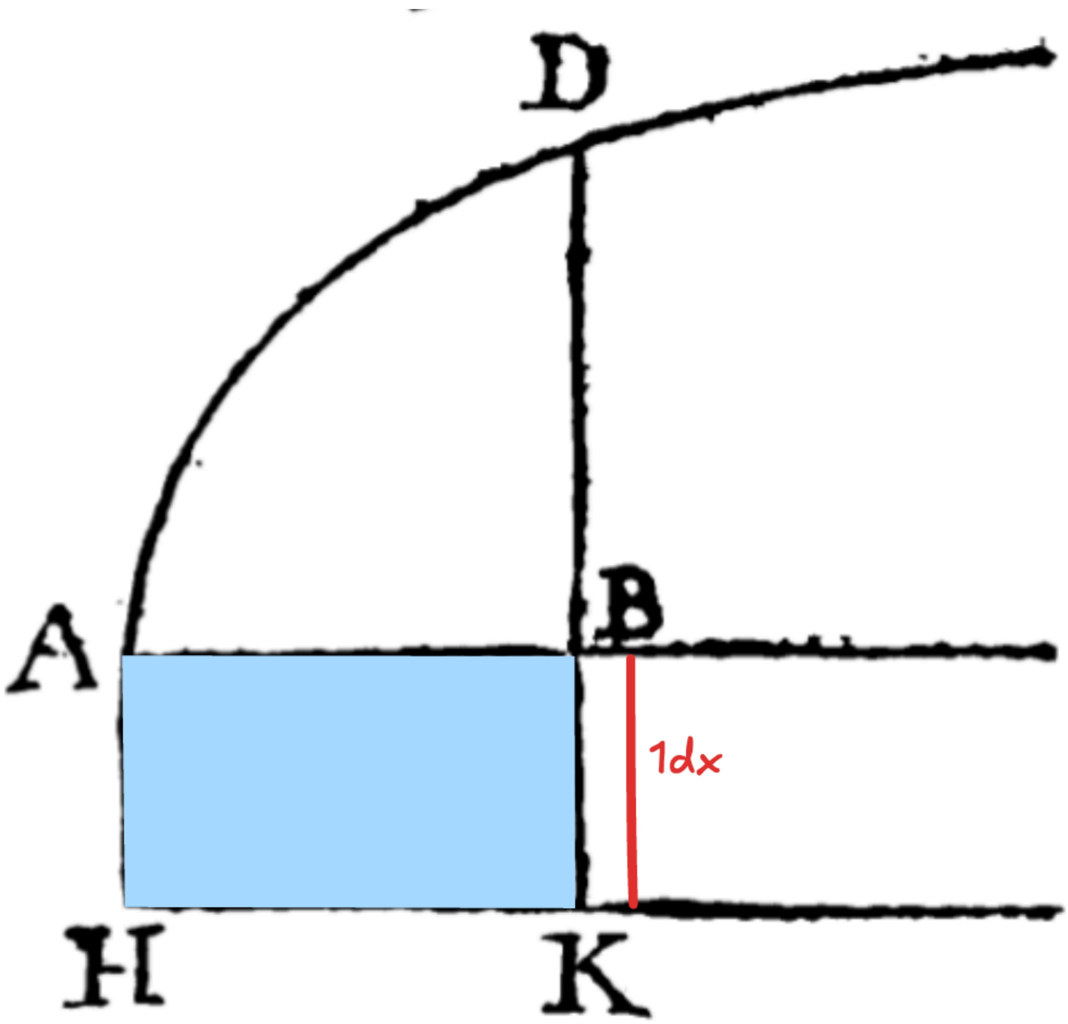


Figure 3: The blue area is "increased continually by the Moment 1". Equivalently, it is increased by the infinitesimal area segment $1dx$

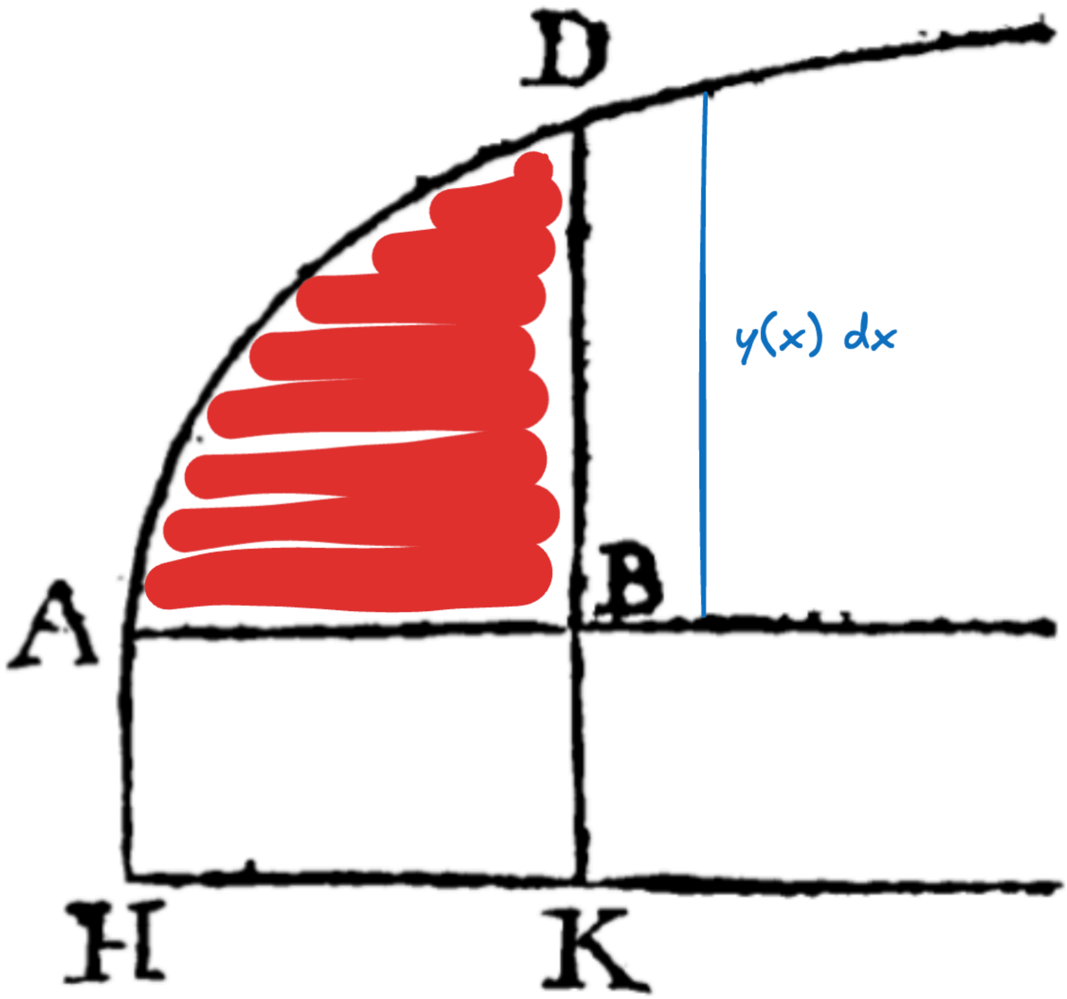


Figure 4: the red area is increased continually by the Moment $y(x)$. Equivalently, it is increased by the infinitesimal area segment $y(x)dx$

> encreaſed ; and that from the Moment BD continually given, you can, by Means of the preceding Rules, inveſtigate the Area ABD deſcribed by it, or compare it with AK (*x*), which is decribed with the Moment 1.

AK is really shorthand for the area $ABKH$. We assume that the 'preceding rules' are the rules of integration and infinite series. When Newton says "investigate the Area ABD described by it, or *compare* it with $AK(x)$", the most reasonable interpretation is that he wants to draw attention to how the area under the curve changes *as a function* of x.

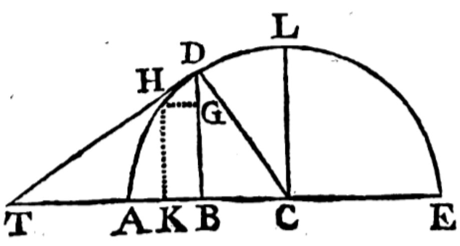
> Now by the ſame Means that the Superficies ABD from it's Moment being at all Times given, is diſcovered by the foregoing Rules, by the like Means may any other Quantity be inveſtigated from it's Moment given in like manner. The Thing will be clearer by an Example.

TODO: < reword for clarity >

If we know the Moment (y) at all times, we can calculate the Superficies ABD (the integral $\int y(x)dx$). It is striking that Newton still uses an x instead of a t for his independent variable, as it is clear from this discussion that the Moment should be viewed as a function of time. Indeed, since x is supposed to vary uniformly with time, knowing y "at all times" is equivalent to knowing the function $y(x)$. "any Quantity may be investigated from it's Moment" is equivalent to saying - "Any Moment (derivative) can be integrated".

TODO: < / reword for clarity >

> *To find the Lengths of Curves.*
>
> 38. Let ADLE be a Circle, the Length of whoſe Arch AD is to be inveſtigated. Draw the Tangent DHT, and having compleated the indefinitely ſmall Rectangle HGBK, and put AE $=1=2$AC, it ſhall be as BK or GH the Moment of the Baſe AB (x) to HD the Moment of the Arch AD : : BT : DT : : BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) : : 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). And ſo



Whereas point § 37 was about finding the areas under curves using line-like 'Moments' (derivatives of areas), § 38 is about finding curves using point-like Moments (infinitesimal line segments).

When Newton writes "BD ($\sqrt{x-xx}$) : DC $\left(\frac{1}{2}\right)$", the brackets should not be taken to mean multiplication. Instead, the statement "$A(B) : C(D)$" actually means "$\frac{A}{C} = \frac{B}{D}$". E: Yes I suppose that's true although I tend to think more simply about these brackets as indicating equality, like he's saying "A (which by the way is equal to B) stands to C (which by the way is equal to D)..." but of course that has the same implications as what you describe. For example, "$1(BK) : \frac{1}{2\sqrt{x-x^2}}(DH)$" actually means "$\frac{BK}{DH} = 2\sqrt{x-x^2}$", the truth of which will become clear in our following discussion, where we expand, using several illustrations involving two sets of similar triangles and one instance of the Pythagorean theorem, what Newton derives above in a pithy one-liner.

Question: What does the "::" mean? Currently I am thinking something like "AND" E: I think what he's saying there is essentially $\frac{BK}{DH} = \frac{BT}{DT} = \frac{BD}{DC}$, so the "::" means "=". V: Yes, makes sense!

We start by noticing that the red and green triangles are similar (see Figure 5) since the two triangles share the angle $\angle BDT$ (or $\angle GDH$) and both triangles contain a right angle ($\angle TBD$ and $\angle HGD$), whence:

$$\frac{DT}{BT} = \frac{DH}{GH} \tag{2}$$

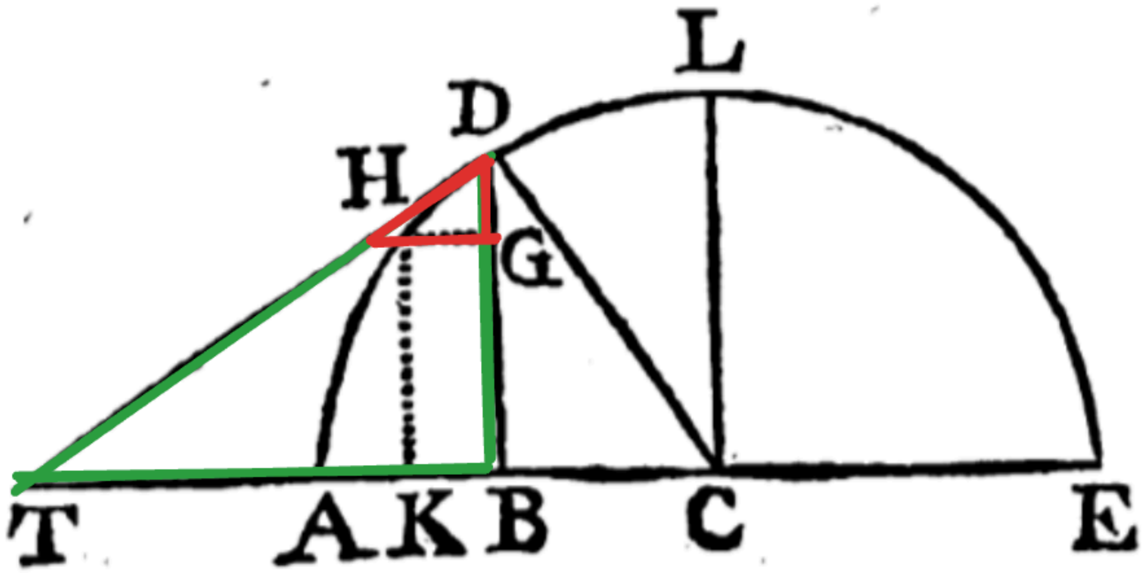


Figure 5: red and green triangles are similar

Next, we notice that the red and green triangles in Figure 6 are also similar, since both contain a right angle ($\angle TBD$ and $\angle DBC$) and $\angle CDB = \angle BTD$. The latter follows from the fact that in $\triangle DBT$, we can see that $90° - \angle BDT = \angle BTD$, and since $\angle CDT = 90°$, we know that $\angle CDB = 90° - \angle BDT$, which we established was equal to $\angle BTD$. TODO: reword for clarity

Therefore:

$$\frac{DT}{BT} = \frac{DC}{BD} \tag{3}$$

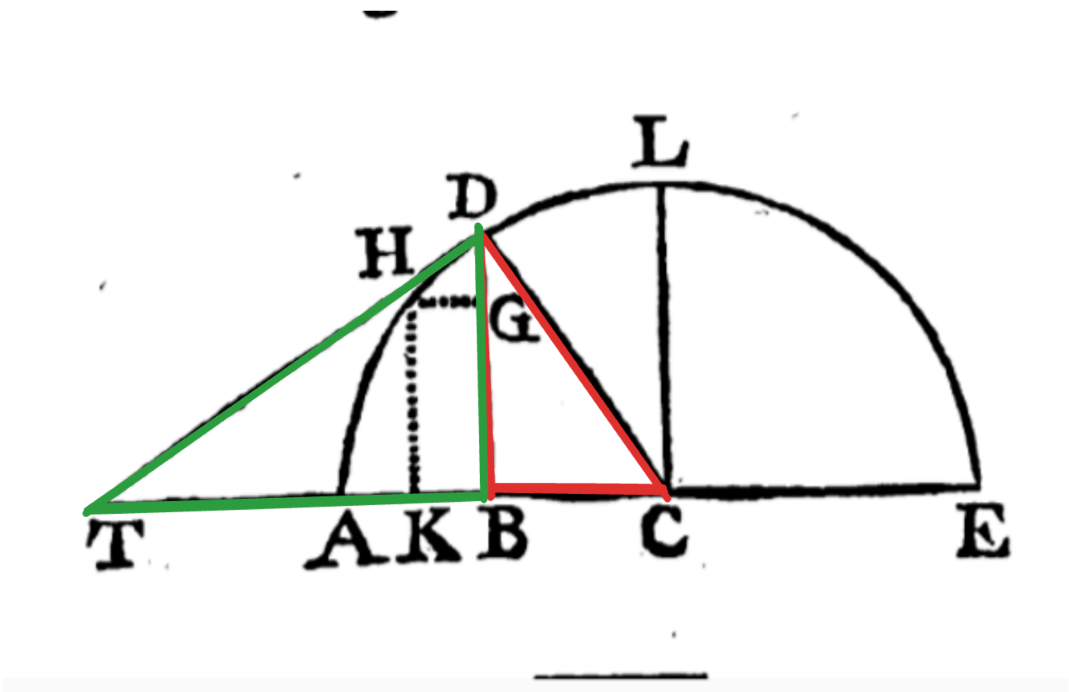


Figure 6: red and green triangles are similar

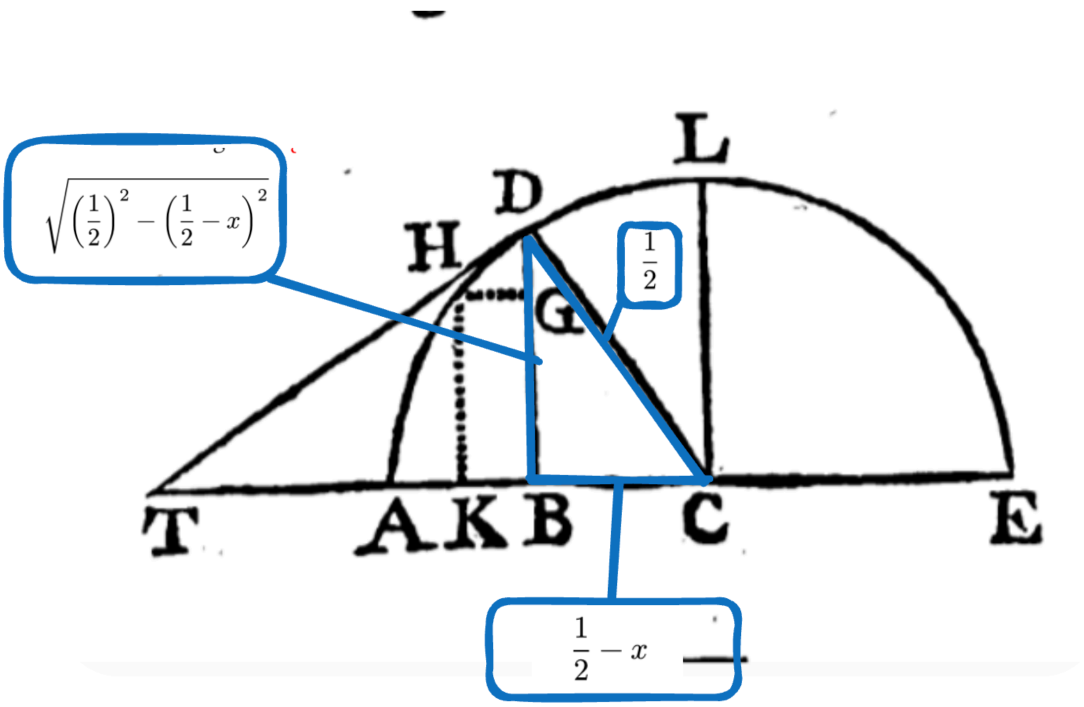


Figure 7: The Pythagorean theorem is used to find the value of the line BD

Next, we use the pythagorean theorem on the blue triangle in Figure 7, to find that:

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} = \sqrt{x - x^2} \tag{4}$$

Finally, by constructing the circle to have a radius of $\frac{1}{2}$, we know that:

Question: Does it have to be $\frac{1}{2}$? Could any value work? What if we had 1? It seems like it matters because in point § 39 he lets it be 1 instead. If $DC = 1$, you simply get $DB = \sqrt{2x - x^2}$, so the series expansion would look slightly different but I think it would still be fine. It is indeed strange that he uses a separate section for a different radius and with a different length he calls x…

$$DC = \frac{1}{2} \tag{5}$$

Combining Equation 2 and Equation 3 to eliminate $\frac{DT}{BT}$ gives:

$$\frac{DH}{GH} = \frac{DC}{BD} \tag{6}$$

Does this not also follow directly from the similarity of triangles in figure 4?

Using Equation 4 and Equation 5 to substitute for DC and BD gives:

$$\frac{DH}{GH} = \frac{1}{2\sqrt{x - x^2}} \tag{7}$$

Now we can rewrite our infinitesimal triangle in a way that will be more recognizable to modern readers, expressing the variable length of the arc AD as $a(x)$: TODO: illustrate d x, d a on a diagram

$$\frac{DH}{GH} = \frac{da}{dx} \tag{8}$$

Therefore:

$$\begin{aligned} \frac{da}{dx} &= \frac{1}{2\sqrt{x - x^2}} \\ a(x) &= \int \left(\frac{1}{2\sqrt{x - x^2}}\right) dx \end{aligned} \tag{9}$$

In Newton's words, $\frac{1}{2\sqrt{x-x^2}}$ is the 'Moment' - i.e *derivative* - of the Arch AD (or $a(x)$ for us).

We can perform the integral by first writing out the series expansion for $\frac{1}{2\sqrt{x-x^2}}$, then integrating term-by-term:

$$\begin{aligned} \frac{da}{dx} &= \frac{1}{2\sqrt{x - x^2}} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \ldots \\ a(x) &= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \ldots \end{aligned} \tag{10}$$

Question: Is this a well-known series expansion? Or is it a well-known expansion modified with some factors? In the modern interpretation text they say that Newton uses the series expansion of $\frac{1}{\sqrt{1-x^2}}$, which he has established in a previous section. If so, maybe we should include a comment/ explanation on that.

> 39. After the ſame Manner by ſuppoſing CB to be x, the Radius CA to be 1, you will find the Arch LD to be $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

Now, instead of $CA = DC = \frac{1}{2}$, we have:

$$DC = 1 \tag{11}$$

and instead of defining $AB = x$ we have $BC = x$, so that by Pythagoras' theorem we obtain

$$BD = \sqrt{1 - x^2}. \tag{12}$$

Combining Equation 6 and Equation 11 we find that

$$\frac{DH}{GH} = \frac{1}{\sqrt{1 - x^2}} \tag{13}$$

which gives

$$a(x) = \int \left(\frac{1}{\sqrt{1 - x^2}} \right) dx. \tag{14}$$

As before, applying a series expansion to the integrand allows us to re-write this expression as

$$a(x) = \int \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \ldots \right) dx \tag{15}$$

and integrating term by term gives

$$a(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots \tag{16}$$

Bibliography

[1] R. Pyke, 'Fluents and Fluxions'. [Online]. Available: https://www.sfu.ca/~rpyke/fluxions.pdf

[2] Newton, *The Application of what has been said to other Problems of the Kind.* 1711. [Online]. Available: https://nx89456.your-storageshare.de/s/Ag2xzMJ8joDXkLd

TODO:

- § 40,
- § 41,
- § 42,
- § 43,
- § 44,
- § 45,
- § 46,
- § 47,

Appendix (to delete)

V: I used these in the diagrams above so they might still be useful when making amendments to the diagrams, but we should delete before handing in

$$\begin{gathered} \frac{1}{2} \\ \frac{1}{2} - x \\ \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} \end{gathered} \tag{17}$$