deadlines
presentation : 9th of June
report: 16th of June

Red text is highly tentative. It may represent an approach that should be substantially revised or deleted

Green text is for TODOs

Question: Text is for as-yet-unresolved questions.

Blue Text is for comments

To modify Victor's illustrations you can here (download and open in excalidraw)

DRAFT
P3 -group 7

Isaac Newton's Infinite Series for the Sine

Emma Ottenhof and Victor Elgersma

15 May 2026

Contents

Glossary

Moment Derivative. For Newton, zero-dimensional Moments generate line segments, one-dimensional Moments generate areas, and two-dimensional Moments generate solids.[1] TODO: illustrate with examples from § 37 and § 38. However, we are somewhat troubled by the fact that when we replace the word "Moment" with derivative in a lot of Newton's writings, statements become trivially true

Superficies Area

Introduction

We provide a line-by-line analysis and modern re-interpretation of a section on the infinite series for the sine, taken from an English translation of Newton's *Analysis by means of Equations with an infinite number of terms*, first published in Latin in 1711.

[1] That Moments have a dimensionality one less than the shapes they generate is clear from the following quote: "40. But it is to be remarked that that Unity which is put for the Moment is a Superficies, when the Question is about Solids; and a Line when about Superficies; and. Point when it is about Lines" (Newton, 1711, p. 336 § 40)

We feel that Newton's diagrams, in which he presents his argument, suffer from the use of too many letters. These letters distract the modern reader from the clarity of his argument. Therefore in addition to a line-by-line commentary we have provided a streamlined modern-recasting, in which we will use letters sparingly, but rely instead on colours to make the argument.

Line by line analysis of Newton sine series

> *The Application of what has been ſaid to other Problems of that Kind.*

> 37. Let ABD be any Curve, and AHKB a Rectangle, whoſe Side AH or BK is Unity :

When considering areas under Curves Newton prefers to consider a 2-dimensional x-axis with a side length unity[2]

> And imagine the Right Line DBK to move uniformly from AH, ſo as to deſcribe the Areas ABD and AK; and that BK (1) is the

> Areas ABD and AK; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreaſed; and that from the Moment BD

It appears that by "Moment" Newton is referring to a value f such that the respective areas increase by $f dx$. This is how we understand the *derivative*. Thus, 1 is the Moment by which the area x is continually increased (in blue, Figure 1. And y is the Moment by which the red area (Figure 2) is continually increased.

[2] We also saw this in Newton's Treatise of the Quadrature of Curves in presentation P1-8

It is clear from our diagram that x is supposed to have the dimensionality of area, whereas y is supposed to have the dimensionality of length. However, we don't believe that it is merely an idiosyncrasy that Newton wants his x-axes to be two dimensional. As we can see from our diagram, there is an obvious parallel between increasing the area of the rectangle by the 'Moment' 1 and increasing the area of the arbitrary curve by the 'Moment' y.

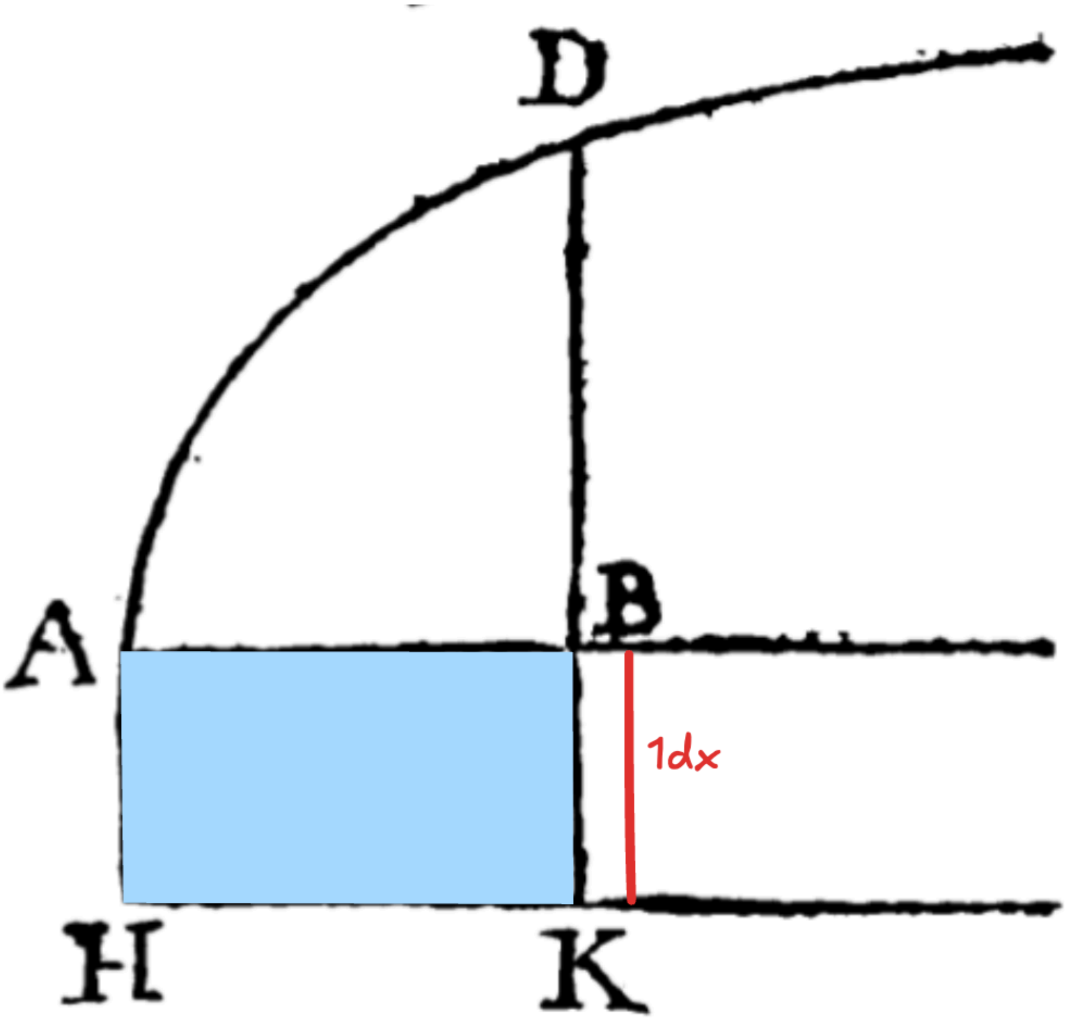


Figure 1: The blue area is "increased continually by the Moment 1". Equivalently, it is increased by the infinitesimal area segment $1dx$

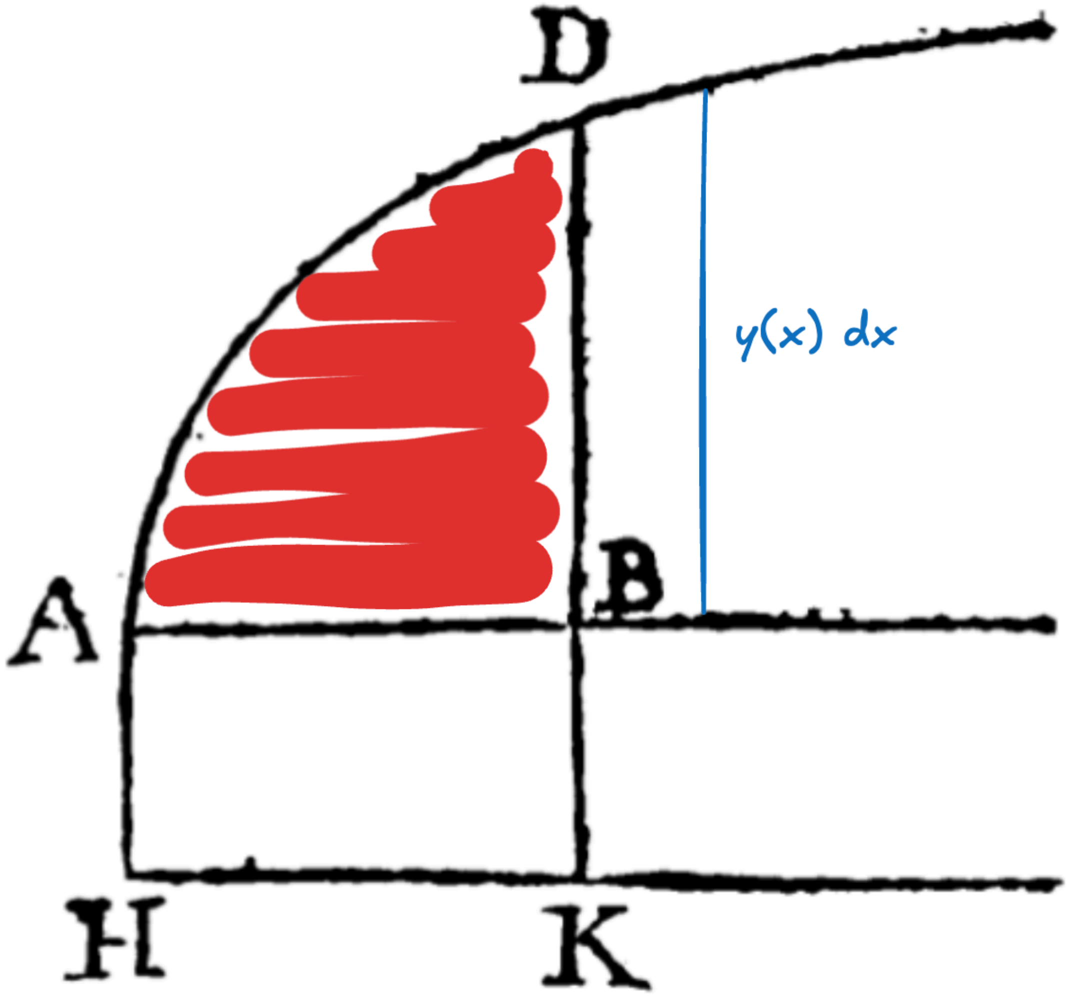


Figure 2: the red area is increased continually by the Moment $y(x)$. Equivalently, it is increased by the infinitesimal area segment $y(x)dx$

> encreafed ; and that from the Moment BD continually given, you can, by Means of the preceding Rules, inveftigate the Area ABD defcribed by it, or compare it with AK (x), which is decribed with the Moment 1.

AK is really shorthand for the area $ABKH$. We assume that the 'preceding rules' are the rules of integration and infinite series. When Newton says "investigate the Area ABD described by it, or *compare* it with $AK(x)$", the most reasonable interpretation is that he wants to draw attention to how the area under the curve changes *as a function* of x.

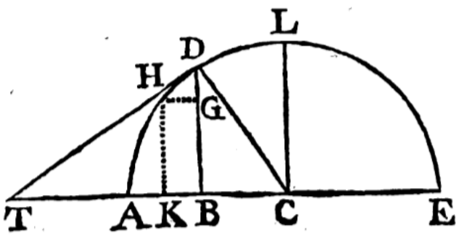
> Now by the ſame Means that the Superficies ABD from it's Moment being at all Times given, is diſcovered by the foregoing Rules, by the like Means may any other Quantity be inveſtigated from it's Moment given in like manner. The Thing will be clearer by an Example.

TODO: < reword for clarity >

If we know the Moment (y) at all times, we can calculate the Superficies ABD (the integral $\int y(x)dx$). It is striking that Newton still uses an x instead of a t for his independent variable, as it is clear from this discussion that the Moment should be viewed as a function of time. Indeed, since x is supposed to vary uniformly with time, knowing y "at all times" is equivalently to knowing the function $y(x)$. "any Quantity may be investigated from it's Moment" is equivalent to saying - "Any Moment (derivative) can be integrated".

TODO: < / reword for clarity >

> *To find the Lengths of Curves.*
>
> 38. Let ADLE be a Circle, the Length of whoſe Arch AD is to be inveſtigated. Draw the Tangent DHT, and having compleated the indefinitely ſmall Rectangle HGBK, and put AE = 1 = 2AC, it ſhall be as BK or GH the Moment of the Baſe AB (x) to HD the Moment of the Arch AD : : BT : DT : : BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) : : 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). And ſo



Whereas point § 37 was about finding the areas under curves using line-like 'Moments' (derivatives of areas), § 38 is about finding curves using point-like Moments (infinitesimal line segments).

When Newton writes "BD ($\sqrt{x - xx}$) : DC ($\frac{1}{2}$)", the brackets should not be taken to mean multiplication. Instead, the statement "$A(B) : C(D)$" actually means "$\frac{A}{C} = \frac{B}{D}$". For example, "$1(BK) : \frac{1}{2\sqrt{x-x^2}}(DH)$" actually means "$\frac{BK}{DH} = 2\sqrt{x - x^2}$", the truth of which will become clear in our following discussion, where we expand, using several illustrations involving two sets of similar triangles and one instance of the Pythagorean theorem, what Newton derives above in a pithy one-liner.

Question: What does the "::" mean? Currently I am thinking something like "AND"

We start by noticing that the red and green triangles are similar (see Figure 3), whence:

TODO: write this as the reciprocal fraction DH/GH and save one step of algebra down the line

$$\frac{BT}{DT} = \frac{GH}{DH} \tag{1}$$

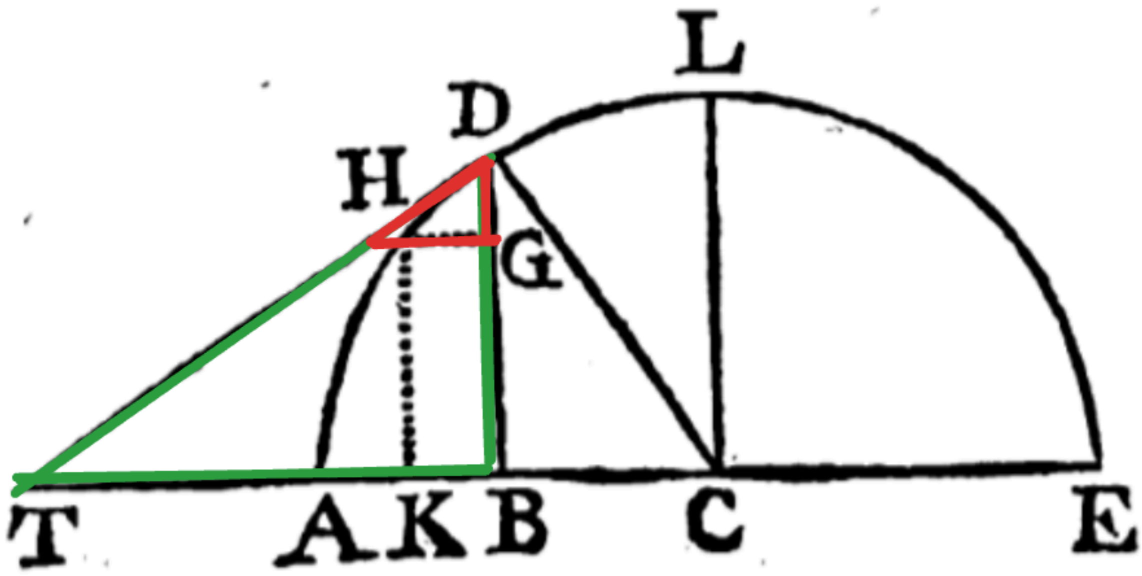


Figure 3: red and green triangles are similar

Next, we notice that the red and green triangles in Figure 4 are also similar, whence:

$$\frac{BT}{DT} = \frac{BD}{DC} \tag{2}$$

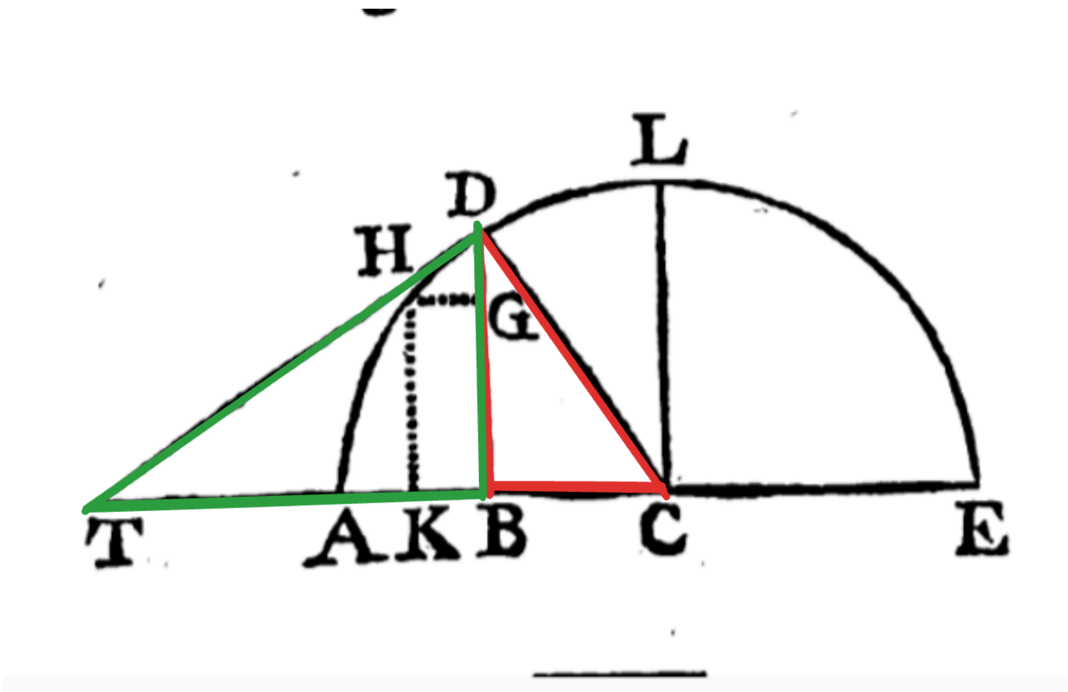


Figure 4: red and green triangles are similar

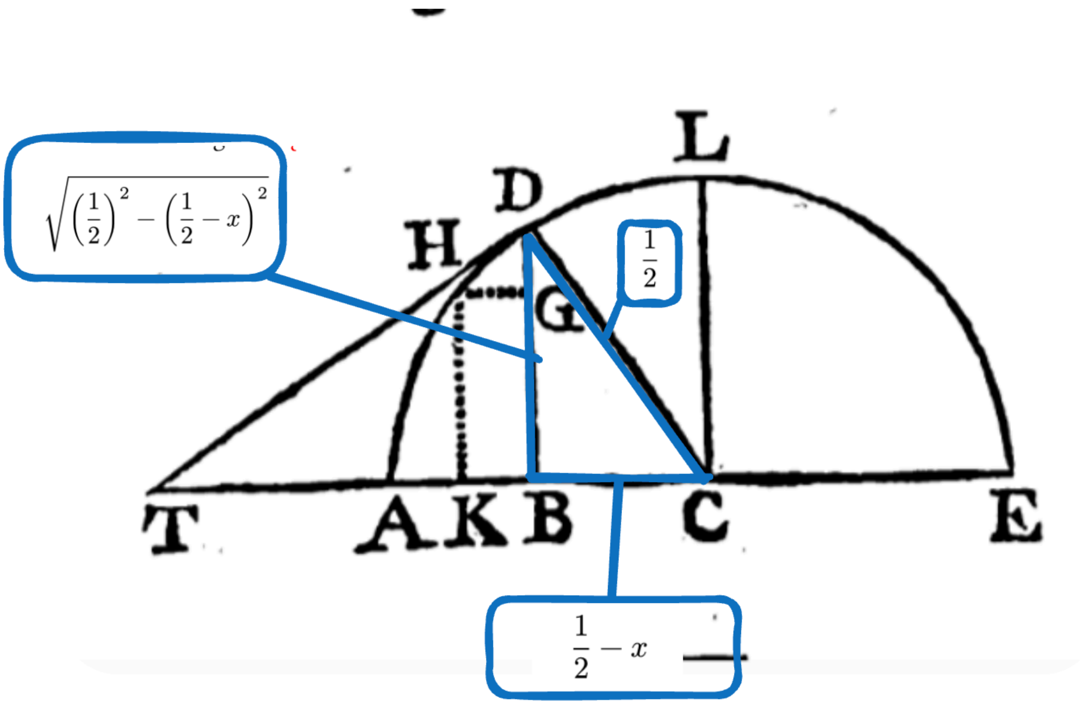


Figure 5: The Pythagorean theorem is used to find the value of the line BD

Next, we use the pythagorean theorem on the blue triangle in Figure 5, to find that:

$$BD = \sqrt{x - x^2} \tag{3}$$

Finally, by constructing the circle to have a radius of $\frac{1}{2}$, we know that:

Question: Does it have to be $\frac{1}{2}$? Could any value work? What if we had 1? It seems like it matters because in point § 39 he lets it be 1 instead.

$$DC = \frac{1}{2} \tag{4}$$

Using Equation 1 and Equation 2 to eliminate $\frac{BT}{DT}$ gives:

$$\frac{GH}{DH} = \frac{BD}{DC} \tag{5}$$

Using Equation 3 and Equation 4 to substitute for BC and DC gives:

$$\frac{GH}{DH} = 2\sqrt{x - x^2} \tag{6}$$

Now we can rewrite our infinitesimal triangle in a more recognizable way: TODO: illustrate d x, d a on a diagram

$$\frac{GH}{DH} = \frac{dx}{da} \tag{7}$$

Therefore:

$$\begin{aligned} \frac{da}{dx} &= \frac{1}{2\sqrt{x - x^2}} \\ a(x) &= \int \left(\frac{1}{2\sqrt{x - x^2}} \right) dx \end{aligned} \tag{8}$$

In Newton's words, $\frac{1}{2\sqrt{x-x^2}}$ is the 'Moment' - i.e *derivative*- of the Arch AD (or $a(x)$ for us).

We can perform the integral by first writing out the series expansion for $\frac{1}{2\sqrt{x-x^2}}$, then integrating term-by-term:

$$\begin{aligned} \frac{da}{dx} &= \frac{1}{2\sqrt{x - x^2}} = \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{4} x^{\frac{1}{2}} + \ldots \\ a(x) &= x^{\frac{1}{2}} + \frac{1}{6} x^{\frac{3}{2}} + \ldots \end{aligned} \tag{9}$$

> "39. After the ſame Manner by ſuppoſing CB to be x, the Radius CA to be 1, you will find the Arch LD to be $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

TODO: Confirm Newton's claim - i.e go through the construction with r=1 and see if we get the same result

Bibliography

Newton. (1711). *The Application of what has been said to other Problems of the Kind.* https://nx89456.your-storageshare.de/s/Ag2xzMJ8joDXkLd

Appendix (to delete)

I used these in the diagrams above so they might still be useful when making amendments to the diagrams, but we should delete before handing in

$$\begin{gathered} \frac{1}{2} \\ \frac{1}{2} - x \\ \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} \end{gathered} \tag{10}$$