On the origin of L'Hôpital's Rule

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    - Bernoulli sells L'Hôpital his rule (Letter dated 22 July 1694)
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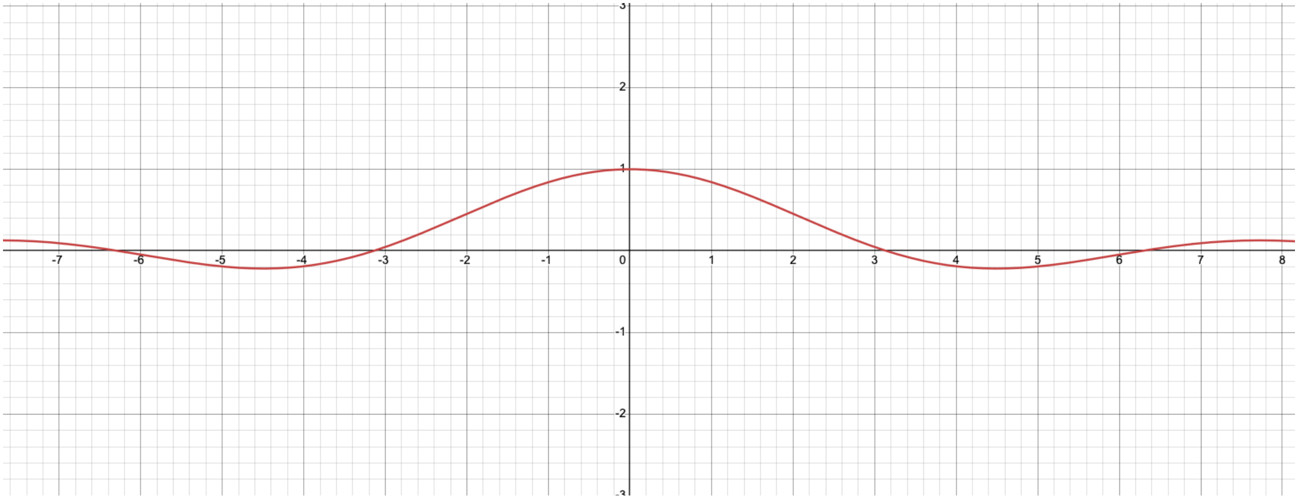
L’Hôpital’s rule today

Applying L'Hôpital's rule to $\frac{\sin(x)}{x}$



Direct Substitution

Try plugging in $x = 0$:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\sin(0)}{0} = \frac{0}{0}$$

😭 Indeterminate form!

Applying L'Hôpital's Rule

Differentiate numerator and denominator:

$$\lim_{x \to 0} \frac{\sin(x)}{x} \underset{\text{L'Hôpital's Rule}}{=} \lim_{x \to 0} \frac{\frac{\mathrm{d}\sin(x)}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}x}}$$

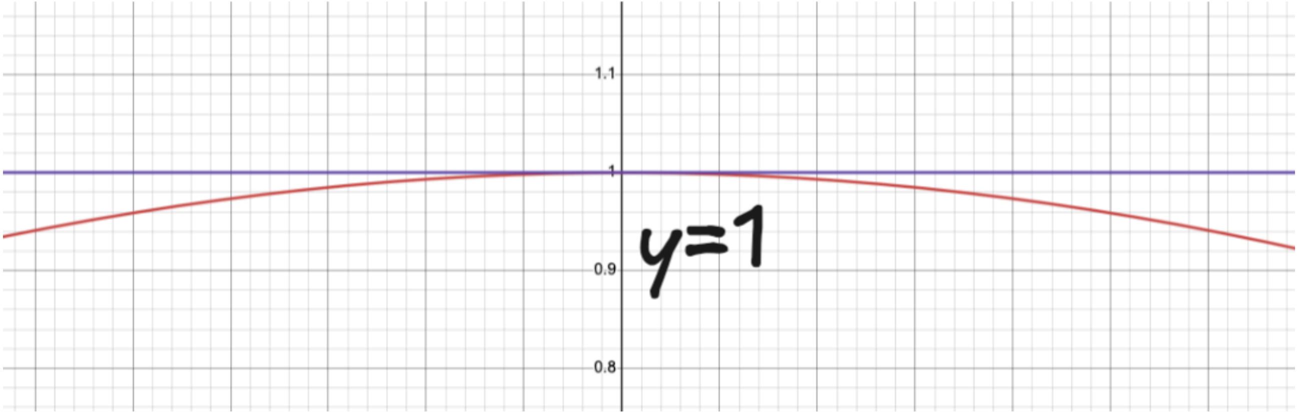
Differentiate & Evaluate

$$\lim_{x\to 0} \frac{\frac{\mathrm{d}\sin(x)}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}x}} = \lim_{x\to 0} \frac{\cos(x)}{1}$$

$$= \frac{\cos(0)}{1} = 1$$

Final Answer

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$



L'Hôpital's Rule in 2026 [1]

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a):

Suppose that:

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

Or

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Assuming the limit on the right exists (or is $\pm\infty$)

Calculus in 1692 vs 2026

	Leibnizian Calculus (1692)	**Modern Calculus (2026)**
...**is about**:	Variable quantities and their differentials	Functions and their derivatives
Differentials are...	Infinitely small quantity; $x \to x + dx$ for the purposes of calculation	Part of notation, eg $\int dx$ or $\frac{d}{dx}$
Derivatives are:	(simply) ratios of differentials	Defined via limits
Functions are	not clearly defined	Formally defined as maps $f : A \to B$
Limits	Not clearly defined	Fundamental
$\sin(x)$ is	Not a standard function; Geometrically understood	Standard Function

1692: Johann Bernoulli tutors the Marquis de L’Hôpital

Postulates I and II of the *Lectiones (1692)* [2]

- *Lectiones de Calculo Differentiales (1692)*

Postulate I

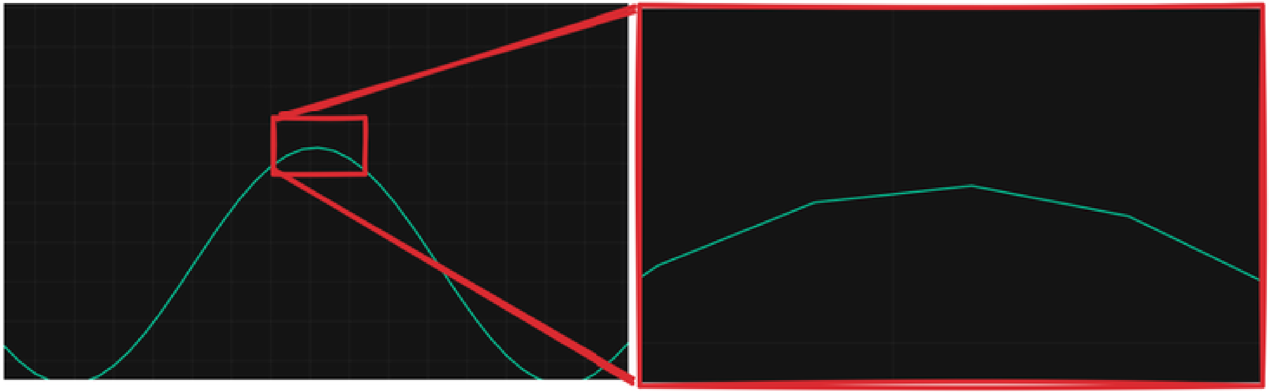
Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

✅ $x \to dx + x$

Postulate II [3]

Postulate II

Any Curved line consists of infinitely many straight lines, each of which is infinitely small



Postulate II [3] (ii)

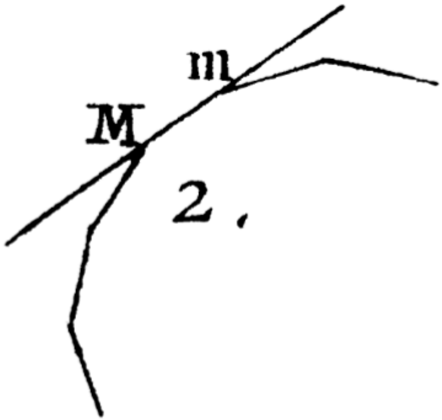


Fig. 2 Postulate II

1693:

Bernoulli's challenge to Varignon

L'Hôpital to Bernoulli (June 27th, 1693):

I saw Mr. Varignon who seems to me to be a very good friend of yours, and he gave me a small piece of paper on which had the following question, which he told me you had sent to him. Let the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

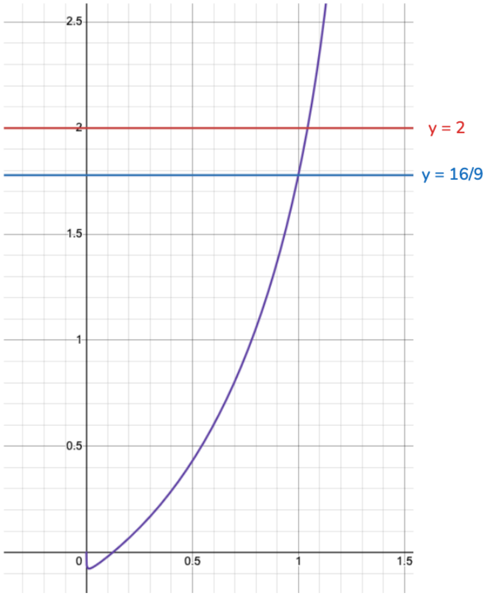
express the nature of a curve whose abscissa is x and whose ordinate is y. We wish to know the value of y when x becomes equal to the constant a. Solution. $y = 2a$, because in this case[22]

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \frac{aa - aa}{a - a} = a + a.$$

L'Hôpital's (incorrect) solution

$$\begin{aligned} y &= \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} && \text{plug in } x = a \\ &= \frac{\sqrt{2a^3a - a^4} - a\sqrt[3]{a^2a}}{a - \sqrt[4]{aa^3}} \\ &= \frac{aa - aa}{a - a} \\ &= \frac{\cancel{(a - a)}(a + a)}{\cancel{a - a}} && \text{difference of two squares} \\ &= \underline{2a} \end{aligned}$$

Plot



Let $a = 1$

$$y = \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - \sqrt[4]{x^3}}$$

L'Hôpital: "Please, Bernoulli, give me the answer"

Letter 15 (September 2, 1693)

I confess that I did not work very hard to solve the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

where $x = a$. Because I see no hope of success, since all the solutions that first present themselves are not correct, I did not want to waste my time unnecessarily, and I'd prefer to learn it from you if you are willing to share it with me. I finish, Sir, by asking you always to love me and to believe me to be entirely yours

The M. De L'Hôpital

Mrs. de L'Hôpital is your servant.

L’Hôpital: “Please, Bernoulli, give me the answer” (ii)

Letter 17: (December 2, 1693):

…But after that, I beg you to remember your promise on the tangent to the tractrices and the solution to the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y \quad \text{where} \quad x = a$$

where I am sure I will see something quite unique and very beautiful, as in all your inventions.

1694

L'Hôpital offers cash for maths

Letter 20: March 17, 1694

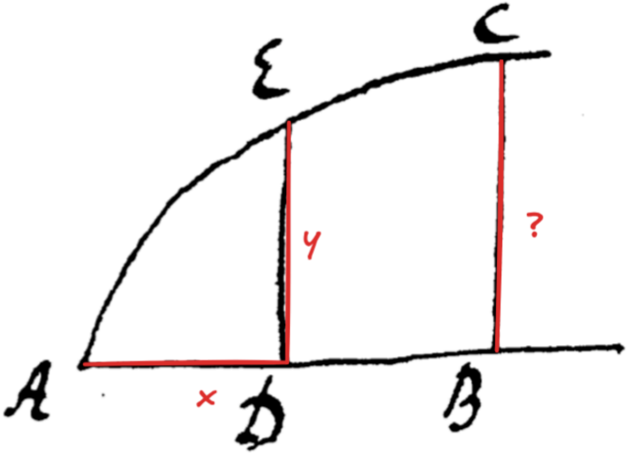
> I will happily to give you a *pension* of three hundred pounds, which will begin the first of January of this present year,
>
> I am not so unreasonable as to demand all of your time for this, but I will ask you at intervals to give me a few hours of your time, to work on what I will ask you and also to communicate your discoveries to me, while asking you at the same time not to share any of them with others.

Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4]

Letter 28: 22 July, 1694:

> *Probl.*[72] Given a curve whose nature is expressed by a fraction equal to y, which in a certain case has the numerator and the denominator equal to zero, we wish to find the value, that is to say the magnitude of the ordinate y.

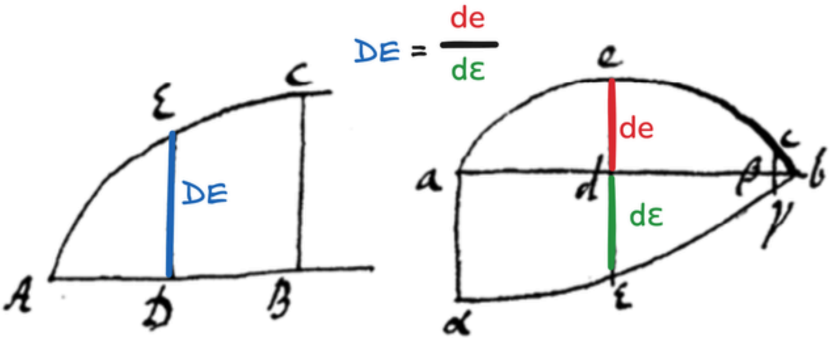
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (ii)



Sol. Let AEC be the given curve, $AD = x$, $DE = y$, AB = a constant such that BC becomes equal to a fraction, the denominator and numerator which are equal to zero.

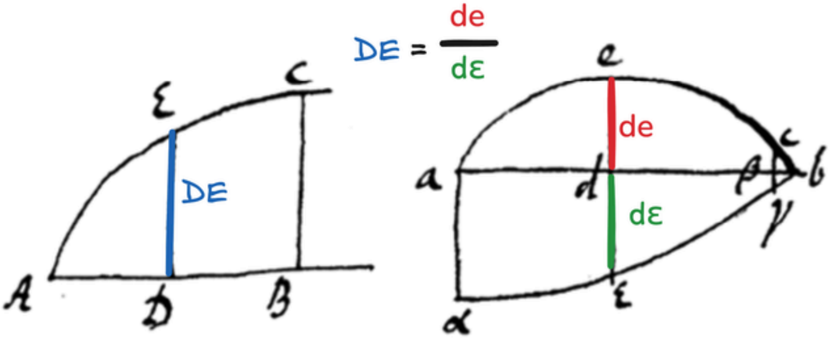
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (iii)

Therefore, to find the magnitude of the ordinate *BC*, I construct on the same axis *adb* two other curves *aeb* and $\alpha\epsilon b$ of such a nature that having taken abscissas equal to *AD* and *ad*, the ordinates *de* are in ratio to the numerator of the general fraction, which expresses the ordinate *DE*, and $d\epsilon$ are in ratio to the denominator of the same fraction.



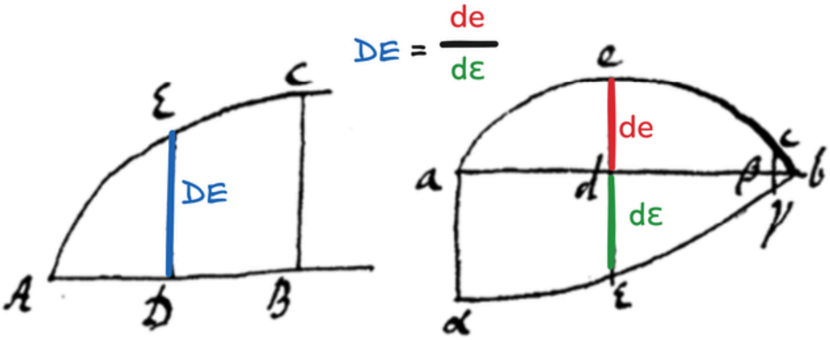
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (iv)

This being done it is clear that *de* divided by $d\epsilon$ may be supposed equal to *DE*. The problem therefore reduces to finding the value of *de* divided by $d\epsilon$ in the case that *ab* is equal to *AB*.



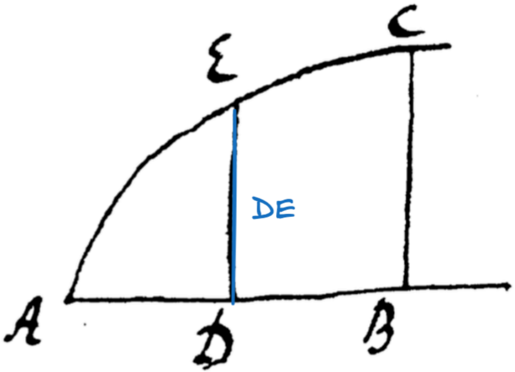
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (v)

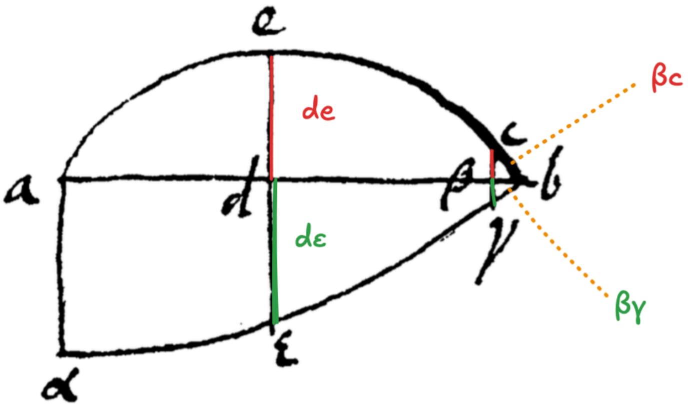
> Now, I see that in this case, *de* and $d\epsilon$ vanish because the two terms of the fraction vanish, and thus the two curves *aeb* and $\alpha\epsilon b$ intersect at the point *b*.



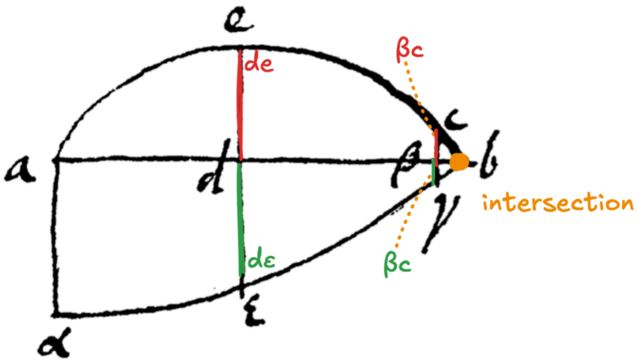
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (vi)

Therefore, we need only take the last differentials βc and $\beta\gamma$, of which the one divided by the other will tell me the magnitude of BC that I seek





Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (vii)



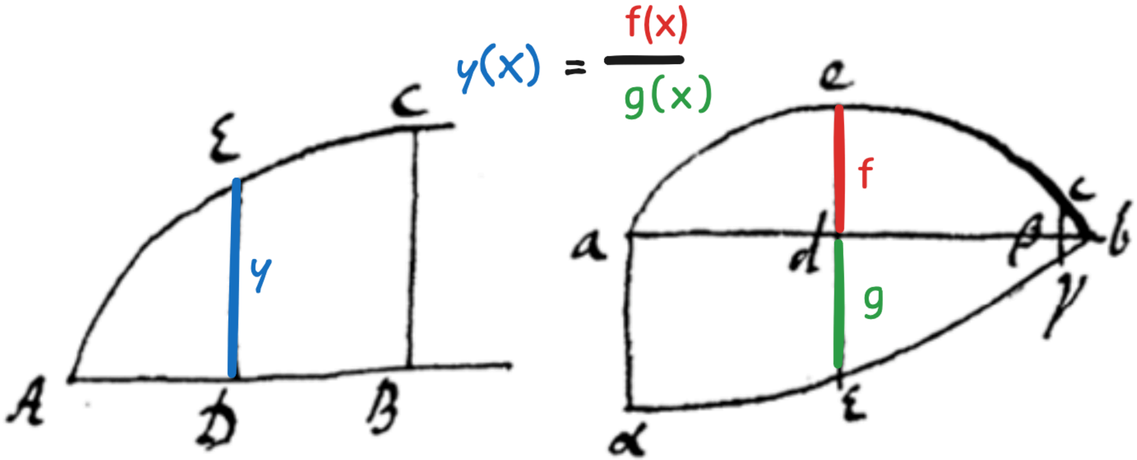
Curves intersect $\Rightarrow \dfrac{de}{d\varepsilon} = \dfrac{\beta c}{\beta\gamma}$?

Postulate I

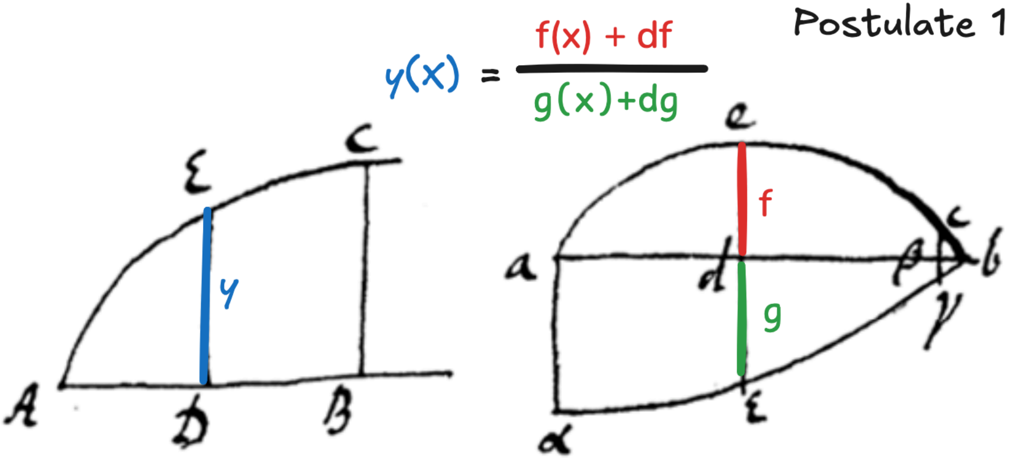
Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

$x \to dx + x$ is ✅ for the purposes of calculation

Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (viii)

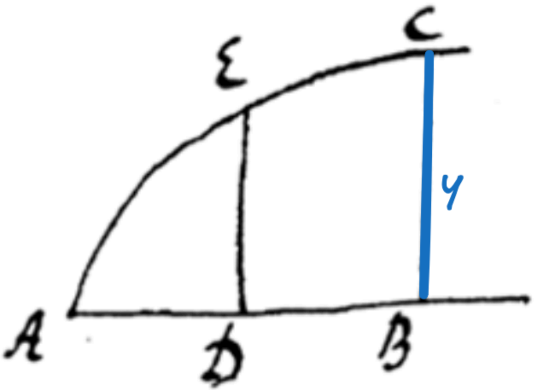


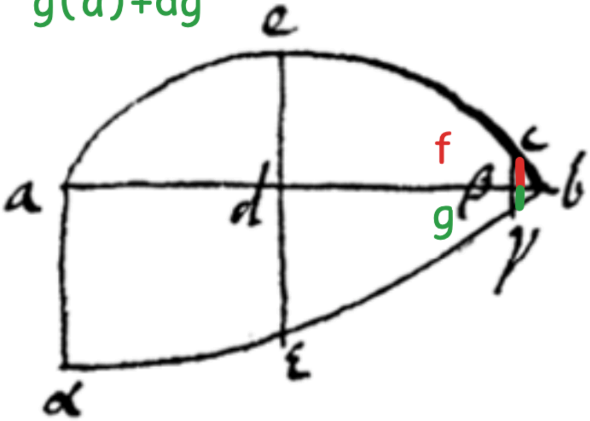
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (ix)



Bernoulli explains L’Hôpital’s rule to L’Hôpital (July 1694) [4] (x)

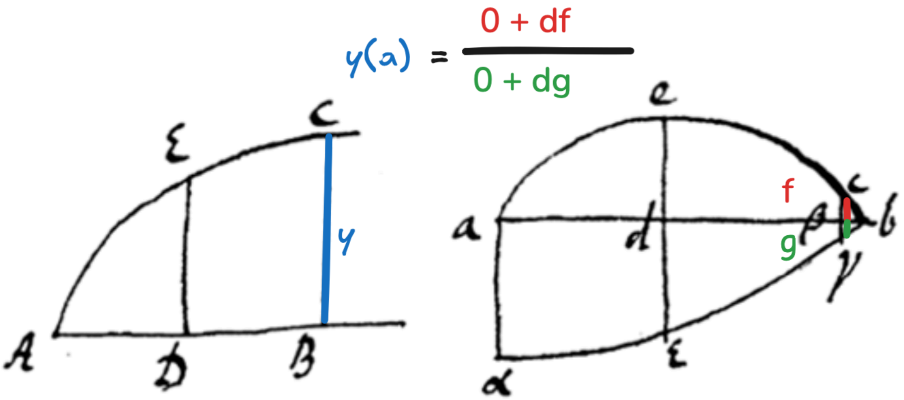
$$y(a) = \frac{f(a) + df}{g(a) + dg}$$



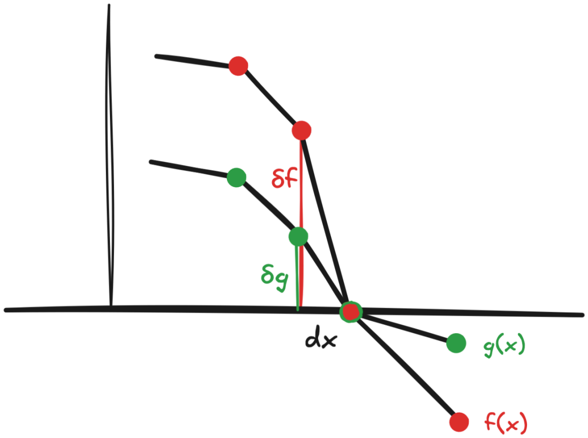


Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) [4] (xi)

Curves intersect:



Summary of Argument



Postulate II

Curves are made of infinitely many infinitesimal line segments.

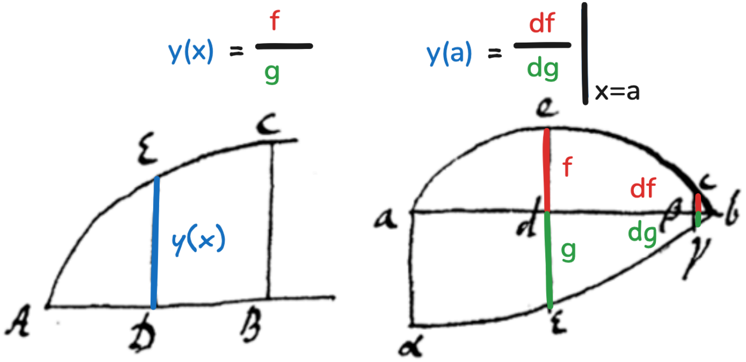
Postulate I

Quantities that change by an infinitesimal remain the same

PI + PII + Curves Intersect and are both 0

$\rightarrow \frac{df}{dg} = \frac{f}{g}$ <u>at the intersection point</u>

Full statement of the Rule (July 1694)



> This is what gives me the following general rule: *To find the value of the ordinate of the given curve in the given case we must divide the differential of the numerator of the general fraction by the differential of the denominator; the quotient, after having made x equal to the supposed AB, will be the magnitude of BC*

Example Problem

> *Example.* The curve ACE has for its equation
>
> $$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} = y.$$

> Thus, if AB is $= a$, we have $BC = \frac{0\,a}{0}$, now we wish to know the true value.

Example Problem (ii)

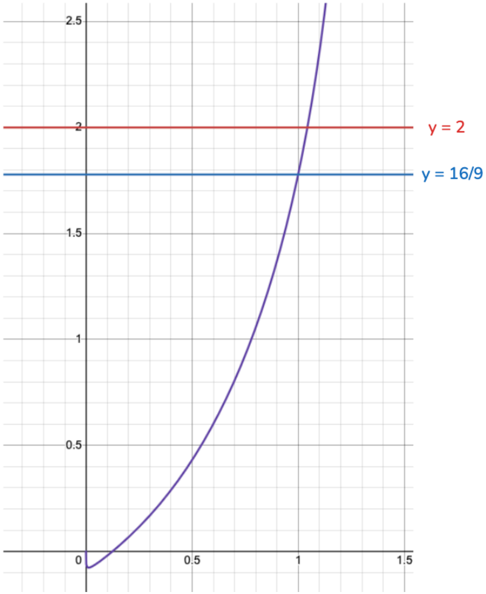
> According to the rule, I take the differential of the numerator $\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}$, which is
>
> $$= \frac{a^3\,dx - 2x^3\,dx}{\sqrt{2a^3x - x^4}} - \frac{a^2\,dx}{3\sqrt[3]{axx}},$$

Example Problem (iii)

having now substituted in the place of x the supposed value a, we find $-\frac{4}{3}a\,dx$ for the first differential and $-\frac{3}{4}\,dx$ for the second one. Therefore,

$$\frac{-\frac{4}{3}a\,dx}{-\frac{3}{4}\,dx} \quad \text{or} \quad \frac{16a}{9} = BC.$$

Example Problem (iv)

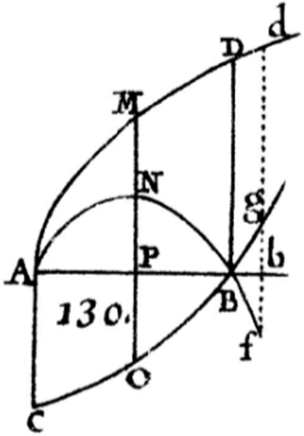


Let $a = 1$

$$y = \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - \sqrt[4]{x^3}}$$

1696

L'Hôpital Ch9 of the *Analyse (1696)*



> […] if we imagine an ordinate bd infinitely close to BD, which meets the curved lines ANB and COB at f and g, then we will have $bd = \frac{AB \times bf}{bg}$, which (see Postulate 1) does not differ from BD. It is therefore only a question of finding the ratio of bg to bf

Postulate I

Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

Fontenelle on Calculus before the *Analyse (1696)* :

> the Geometry of the Infinitely small was still nothing but a kind of Mystery, and, so to speak, a Cabalistic Science shared among five or six people. They often gave their Solutions in the Journals without revealing the Method that produced them, and even when one could discover it, it was only a few feeble rays of this Science that had escaped, and the clouds immediately closed again.
>
> — Fontenelle, 1708

Bernoulli complains to Varignon:

> to speak frankly, Mr. de L'Hôpital had no other part in the production of this book than to have translated into French the material that I gave him, for the most part, in Latin
>
> — Bernoulli to Varignon, 1707

Summary

	L'Hôpital's Theorem 2026	L'Hôpital's Theorem 1696
... is about	The **limit** of a quotient of two functions $\frac{f(x)}{g(x)}$	The **actual value** of the quotient.
...uses	**derivatives** defined via **limits**: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.	ratios of **differentials** ($\frac{df}{dg}$) treated as tiny line segments.
... is proved using	The **Mean Value Theorem** (real analysis).	**Postulates 1 and 2** of the *Lectiones* and some geometry

References

[1] J. Stewart, *Calculus: Early Transcendentals*, 8th ed. Belmont, CA, USA: Cengage Learning, 2015.

[2] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, "Bernoulli's Lectiones de Calculo Differentialis," in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 187–231. doi: 10.1007/978-3-319-17115-9_11.

[3] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, "In Which We Give the Rules of This Calculus," in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 1–9. doi: 10.1007/978-3-319-17115-9_1.

[4] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, "Selected Letters from the Correspondence Between the Marquis de L'Hôpital and Johann Bernoulli," in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 233–293. doi: 10.1007/978-3-319-17115-9_12.