

On the origin of L'Hôpital's theorem

Or: Improving our understanding of Leibnizian calculus by reading L'Hôpital's *Analyse* (1696), Bernoulli's *Lectiones* (1692), and selected letters (1693-94) from the L'Hôpital - Bernoulli correspondence

March 29, 2026

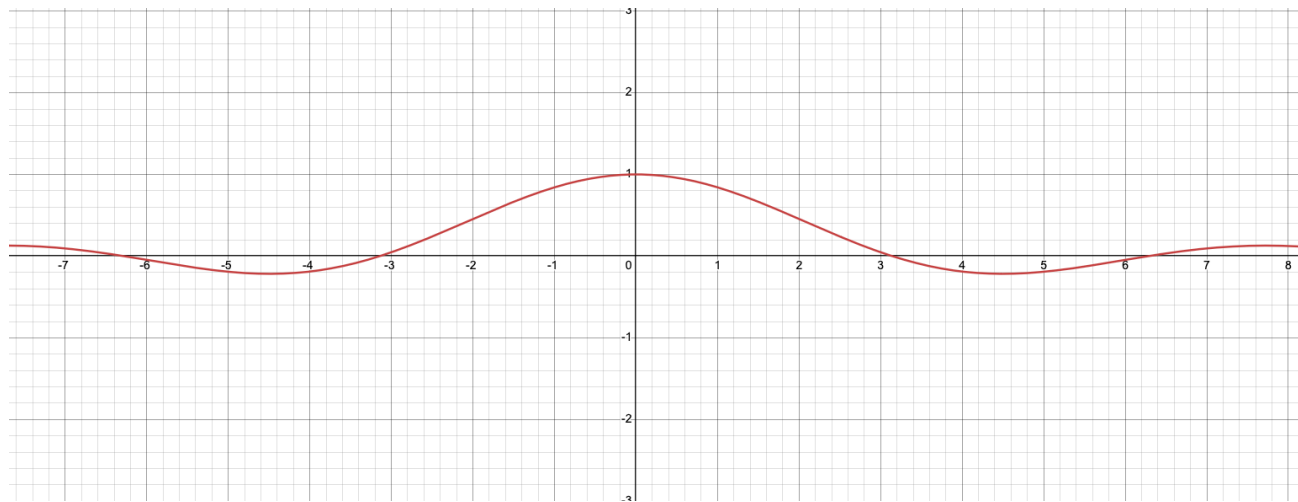
Victor Elgersma & Vera Belde

1 Contents

- Reminder: L'Hôpital's rule today
- 1692: The Marquis de L'Hôpital's receives private tuition from Johann Bernoulli
 - Postulates I and II of Bernoulli's *Lectiones de Calculo Differentialibus* (1692)
- 1693:
 - Bernoulli's challenges Varignon to evaluate an expression of indeterminate form $\frac{0}{0}$
 - L'Hôpital proposes an incorrect solution
- 1694:
 - L'Hôpital pays Bernoulli for exclusive rights to publish his discoveries (March 1694)
 - Bernoulli sells L'Hôpital his rule (Letter dated 22 July 1694)
- 1696:
 - L'Hôpital publishes *Analyse des infiniment petits, pour l'intelligence des lignes courbes* (1696)
 - L'Hôpital's rule appears in Chapter 9.
- Conclusion
 - Significance of the *Analyse* (1696)
 - Bernoulli's sellers remorse.

2 L'Hôpital's rule today

Applying L'Hôpital's rule to $\frac{\sin(x)}{x}$



Direct Substitution

Try plugging in $x = 0$:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{\sin(0)}{0} = \frac{0}{0}$$

🤔 Indeterminate form!

Applying L'Hôpital's Rule

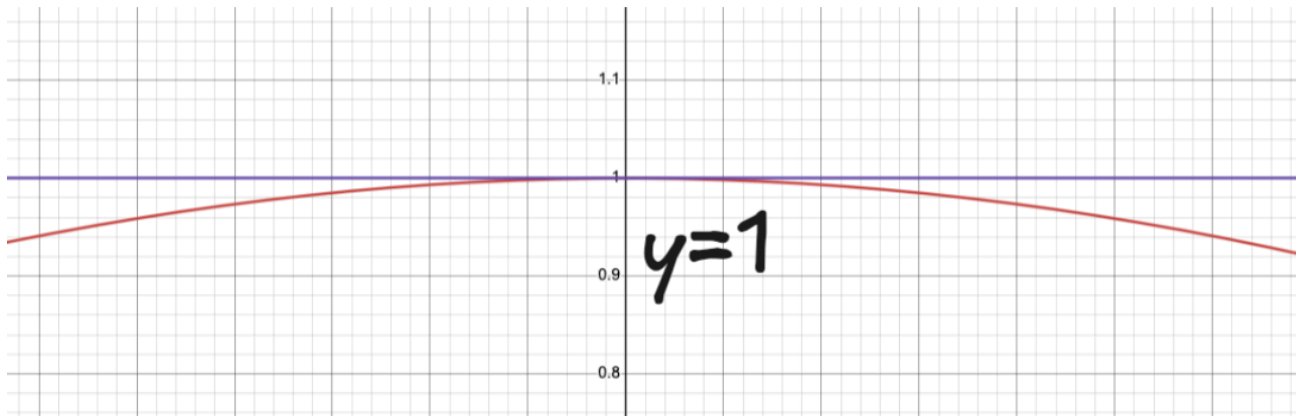
Differentiate numerator and denominator:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} \stackrel{\text{L'Hôpital's Rule}}{=} \lim_{x \rightarrow 0} \frac{\frac{d \sin(x)}{dx}}{\frac{dx}{dx}}$$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\frac{d \sin(x)}{dx}}{\frac{dx}{dx}} &= \lim_{x \rightarrow 0} \frac{\cos(x)}{1} \\ &= \frac{\cos(0)}{1} = 1\end{aligned}$$

Final Answer

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$



L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or that

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Aspect	Leibnizian Calculus (1692)	Modern Calculus (2026)
Fundamentals	Equations and differentials	Functions and derivatives
Differentials (dx)	Infinitely small quantity; $x \rightarrow x + dx$ for the purposes of calculation	-
Derivatives ($\frac{df}{dx}$ or f')	Ratio of differentials	Defined via limits
Functions $f(x)$	-	Formally defined as maps $f : A \rightarrow B$
Limits	-	Fundamental
$\sin(x)$	-	Standard Function

3 1692: Johann Bernoulli tutors the Marquis de L'Hôpital


Postulates I and II of the *Lectiones* (1692) [2]

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- *Lectiones de Calculo Differentiales* (1692)

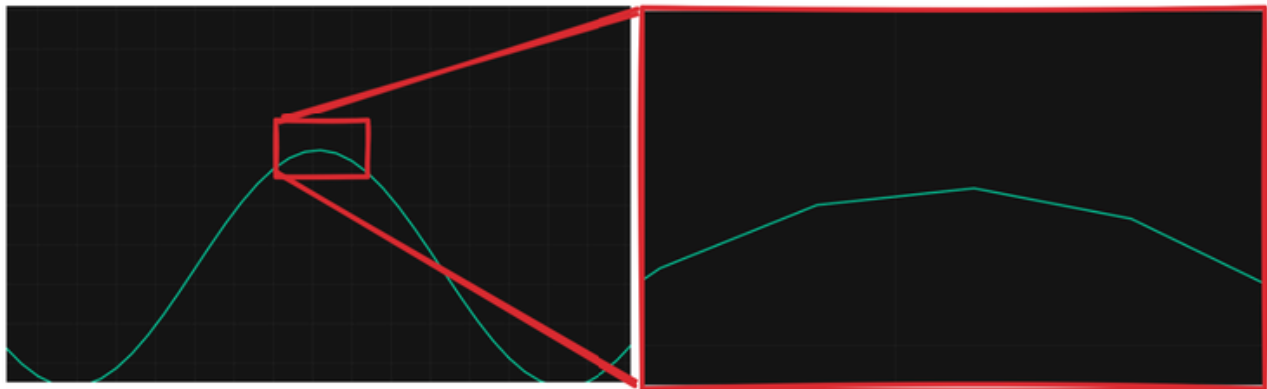
Postulate I

Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

$x \rightarrow dx + x$ is  for the purposes of calculation

Postulate II

Any Curved line consists of infinitely many straight lines, each of which is infinitely small



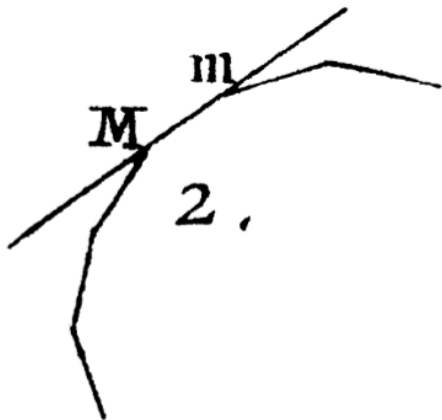


Fig. 2 Postulate II

4 1693:

Bernoulli's challenge to Varignon

L'Hôpital to Bernoulli (June 27th, 1693):

I saw

Mr. Varignon who seems to me to be a very good friend of yours, and he gave me a small piece of paper on which had the following question, which he told me you had sent to him. Let the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

express the nature of a curve whose abscissa is x and whose ordinate is y . We wish to know the value of y when x becomes equal to the constant a . Solution. $y = 2a$, because in this case²²

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \frac{aa - aa}{a - a} = a + a.$$

L'Hôpital's (incorrect) solution

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

plug in $x = a$

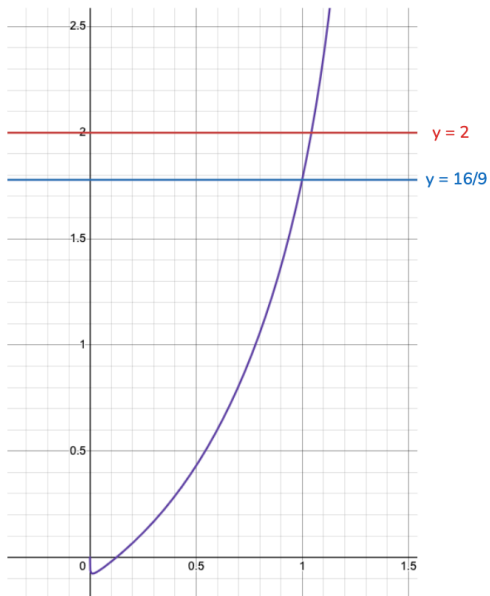
$$= \frac{\sqrt{2a^3a - a^4} - a\sqrt[3]{a^2a}}{a - \sqrt[4]{aa^3}}$$

$$= \frac{aa - aa}{a - a}$$

$$= \frac{(\cancel{a-a})(a+a)}{\cancel{a-a}}$$

difference of two squares

$$= \underline{2a}$$



Let $a = 1$

$$y = \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - \sqrt[4]{x^3}}$$

L'Hôpital: "Please, Bernoulli, give me the answer"

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Letter 15 (September 2, 1693)

I confess that I did not work very hard to solve the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y$$

where $x = a$. Because I see no hope of success, since all the solutions that first present themselves are not correct, I did not want to waste my time unnecessarily, and I'd prefer to learn it from you if you are willing to share it with me. I finish, Sir, by asking you always to love me and to believe me to be entirely yours

The M. De L'Hôpital

Mrs. de L'Hôpital is your servant.

L'Hôpital: "Please, Bernoulli, give me the answer" (ii)

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Letter 17: (December 2, 1693):

...But after that, I beg you to remember your promise on the tangent to the tractrices and the solution to the equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = y \quad \text{where } x = a$$

where I am sure I will see something quite unique and very beautiful, as in all your inventions.

5 1694

L'Hôpital offers cash for the exclusive right to publish his discoveries^{23/44}

Letter 20: March 17, 1694

I will happily to give you a *pension*⁴¹ of three hundred pounds, which will begin the first of January of this present year, and I will send you two hundred pounds for the first half of the year because of the journals you have sent me, and the other half year will be of one hundred fifty pounds, and so on in the future. I promise you to increase this stipend shortly, which I well understand to be very modest, and it will be as soon as my affairs are somewhat straightened out, and I can enjoy the succession, because at present I haven't yet touched anything. I am not so unreasonable as to demand all of your time for this, but I will ask you at intervals to give me a few hours of your time, to work on what I will ask you and also to communicate your discoveries to me, while asking you at the same time not to share any of them with others. I even ask you not to send here to Mr. Varignon, nor to others, any copies of the writings you have left with me; if they should become public I would not be at all pleased. Answer me on all of this and believe me, Sir, to be entirely yours

the M. de L'Hôpital

Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694)

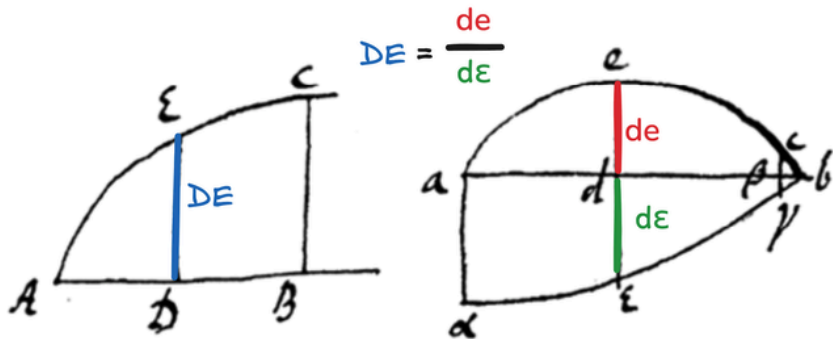
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Letter 28: March 17, 1694:

*Probl.*⁷² Given a curve whose nature is expressed by a fraction equal to y , which in a certain case has the numerator and the denominator equal to zero, we wish to find the value, that is to say the magnitude of the ordinate y .

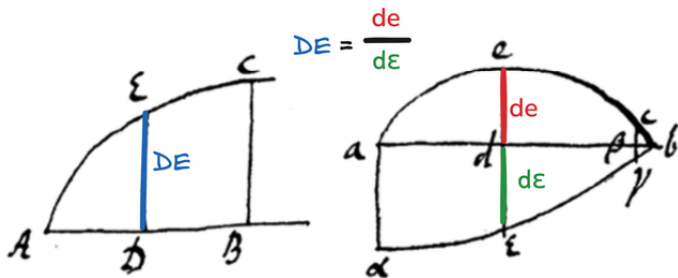
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) (ii)

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The problem therefore reduces to finding the value of de divided by $d\epsilon$ in the case that ab is equal to AB .

Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) (iii) 26 / 44




Now, I see that in this case, de and $d\epsilon$ vanish because the two terms of the fraction vanish, and thus the two curves aeb and αeb intersect at the point b .

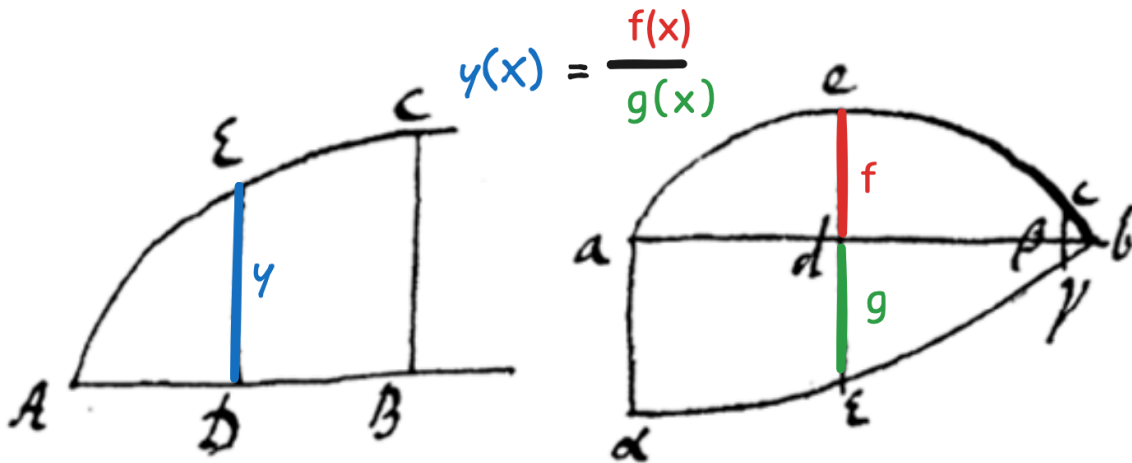
Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) (iv)

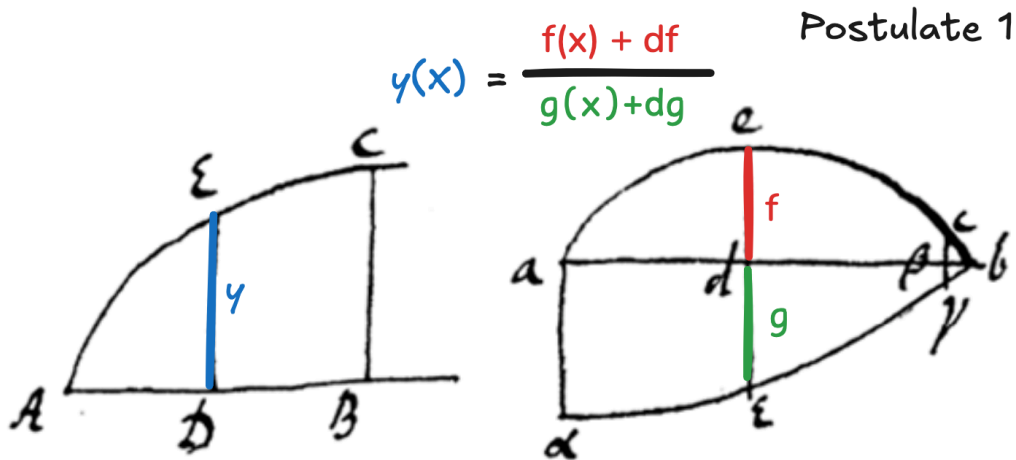
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Postulate I

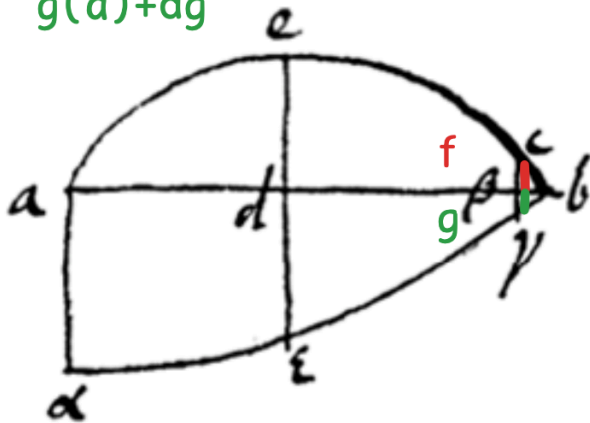
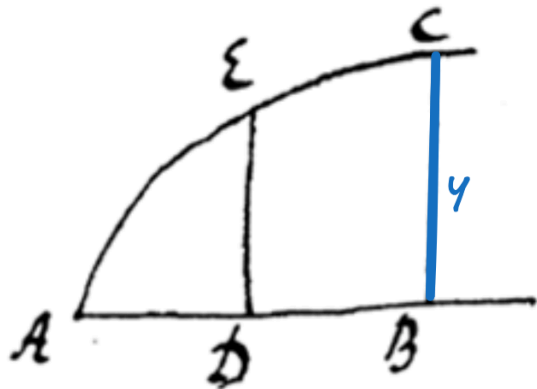
Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

$x \rightarrow dx + x$ is  for the purposes of calculation



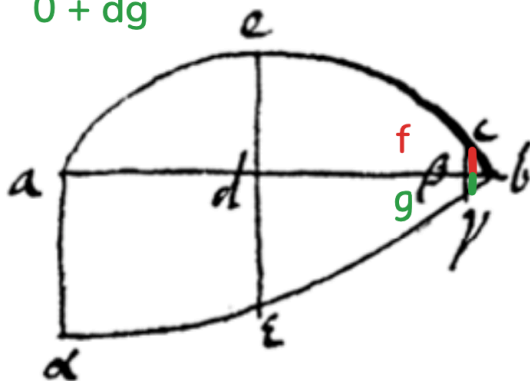
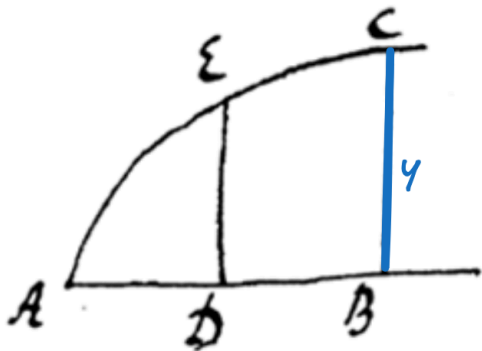


$$y(a) = \frac{f(a) + df}{g(a) + dg}$$



Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) (viii) 31 / 44

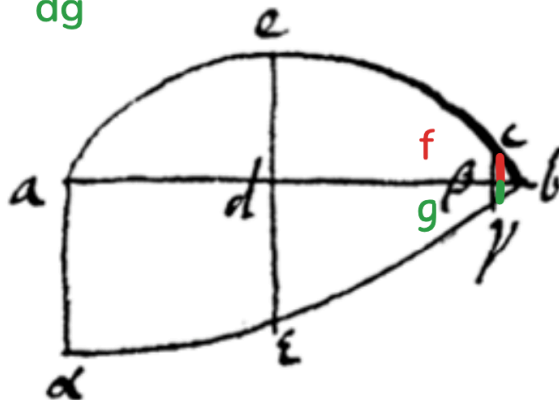
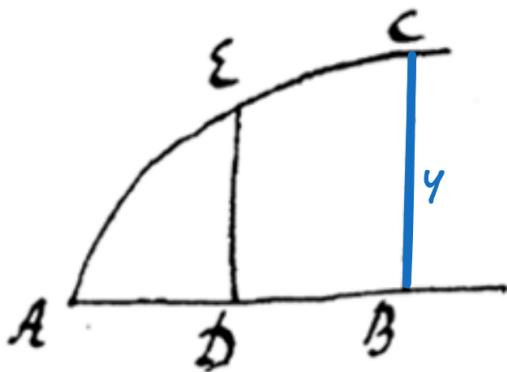
$$y(a) = \frac{0 + df}{0 + dg}$$



Bernoulli explains L'Hôpital's rule to L'Hôpital (July 1694) (ix)

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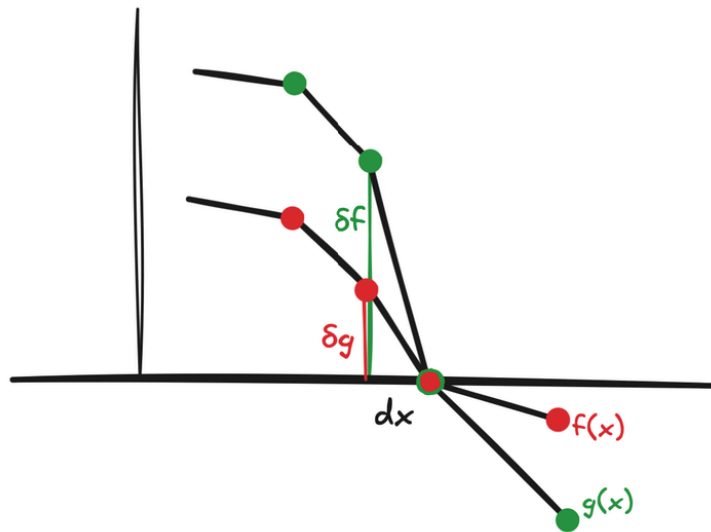
$$y(a) = \frac{df}{dg}$$



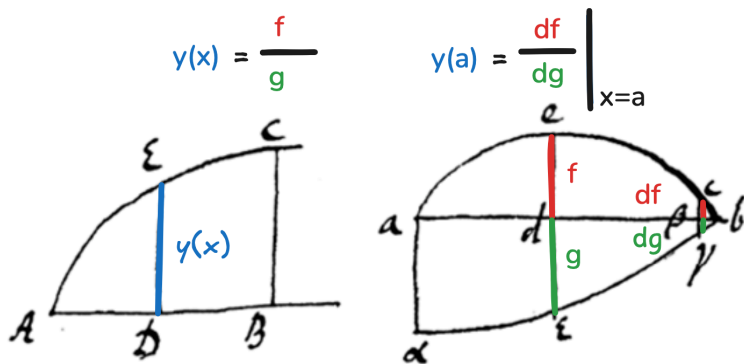
Therefore, we need only take the last differentials βc and $\beta \gamma$, of which the one divided by the other will tell me the magnitude of BG that I seek



The 'last differentials':



Full statement of the Rule (July 1694)



This is what gives me the following general rule: *To find the value of the ordinate of the given curve in the given case we must divide the differential of the numerator of the general fraction by the differential of the denominator; the quotient, after having made x equal to the supposed AB , will be the magnitude of BC*

Example. The curve *ACE* has for its equation

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} = y.$$

Thus, if AB is $= a$, we have $BC = \frac{0a}{0}$, now we wish to know the true value.

Example Problem (ii)

According to the rule, I take the differential of the numerator $\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}$, which is

$$= \frac{a^3 dx - 2x^3 dx}{\sqrt{2a^3x - x^4}} - \frac{a^2 dx}{3\sqrt[3]{aax}}$$

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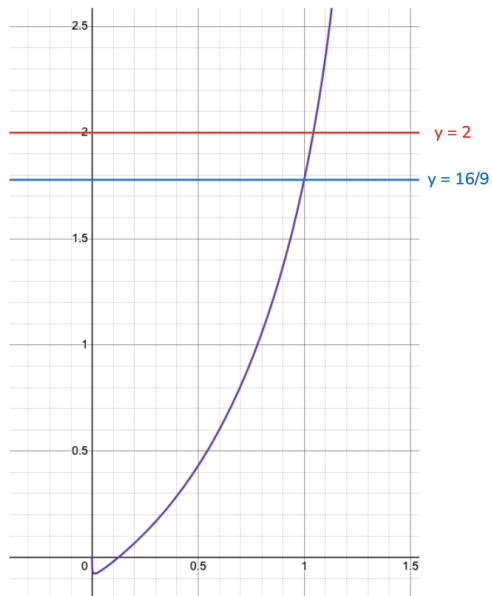
Example Problem (iii)

having now substituted in the place of x the supposed value a , we find $-\frac{4}{3}a dx$ for the first differential and $-\frac{3}{4} dx$ for the second one. Therefore,

$$\frac{-\frac{4}{3}a dx}{-\frac{3}{4} dx} \quad \text{or} \quad \frac{16a}{9} = BC.$$

Example Problem (iv)

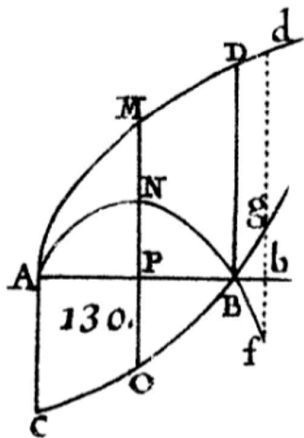
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Let $a = 1$

$$y = \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - \sqrt[4]{x^3}}$$

6 1696



[...] if we imagine an ordinate bd infinitely close to BD , which meets the curved lines ANB and COB at f and g , then we will have $bd = \frac{AB \times bf}{bg}$, which (see Postulate 1) does not differ from BD . It is therefore only a question of finding the ratio of bg to bf

Postulate I

Quantities that decrease or increase by an infinitely small quantity neither decrease nor increase

the Geometry of the Infinitely small was still nothing but a kind of Mystery, and, so to speak, a Cabalistic Science shared among five or six people. They often gave their Solutions in the Journals without revealing the Method that produced them, and even when one could discover it, it was only a few feeble rays of this Science that had escaped, and the clouds immediately closed again.

— Fontenelle, 1708

Bernoulli complains to Varignon:

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to speak frankly, Mr. de L'Hôpital had no other part in the production of this book than to have translated into French the material that I gave him, for the most part, in Latin

— Bernoulli to Varignon, 1707

- [1] J. Stewart, *Calculus: Early Transcendentals*, 8th ed. Belmont, CA, USA: Cengage Learning, 2015.
- [2] R. E. Bradley, S. J. Petrilli, and C. E. Sandifer, “Bernoulli's Lectiones de Calculo Differentialis,” in *L'Hôpital's Analyse des infiniments petits: An Annotated Translation with Source Material by Johann Bernoulli*, Cham: Springer International Publishing, 2015, pp. 187–231. doi: [10.1007/978-3-319-17115-9_11](https://doi.org/10.1007/978-3-319-17115-9_11).