P1 – Group 7

Line-by-Line Commentary

Discovering the Binomial Theorem

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Unpublished Manuscript CUL Add. MS 3958.3, f. 72.[1]

In this page from a young Newton's handwritten notes, we find the integrals of $\frac{1}{x}$ and $\sqrt{1-x^2}$ as power series, which Newton 'intuits' by extrapolating the pattern created by a broader class of easier integrals. In his second letter to Leibniz, he explains that this is how he first stumbled upon these series, but notes that polynomial division and the root extraction algorithm provide a firmer foundation.

What follows is not a proof in a sense that a modern mathematician would recognize[2], but we do think it is pretty convincing.

We first turn to the hyperbola.

Integral of the Hyperbola

> If *lab* is an Hyperbola; *cde*, *ck* its Asymptotes, *a* its vertex, and *cag* its axis; if *adck* is a square and *he* is parallel to *ad*, and *cd* $= 1$, and *de* $= x$, then *be* $= \frac{1}{1+x}$. If also, *ef* $= 1$, *eg* $= 1 + x$, *eh* $= 1 + 2x + x^2$ etc. (the progression continued is $1 + 3x + 3xx + x^3$, $1 + 4x + 6x^2 + 4x^3 + x^4$, $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ etc).

This boils down to "draw the curves $y = \frac{1}{x'}$, $y = 1$, $y = x'$, $y = x'^2$ and $y = x'^3$ in the $x' - y$ plane.

So the origin of the coordinate system is at d. Better to keep this sensible choice in your modern plot as well.

For each of these curves he notes the values at $x' = x + 1$

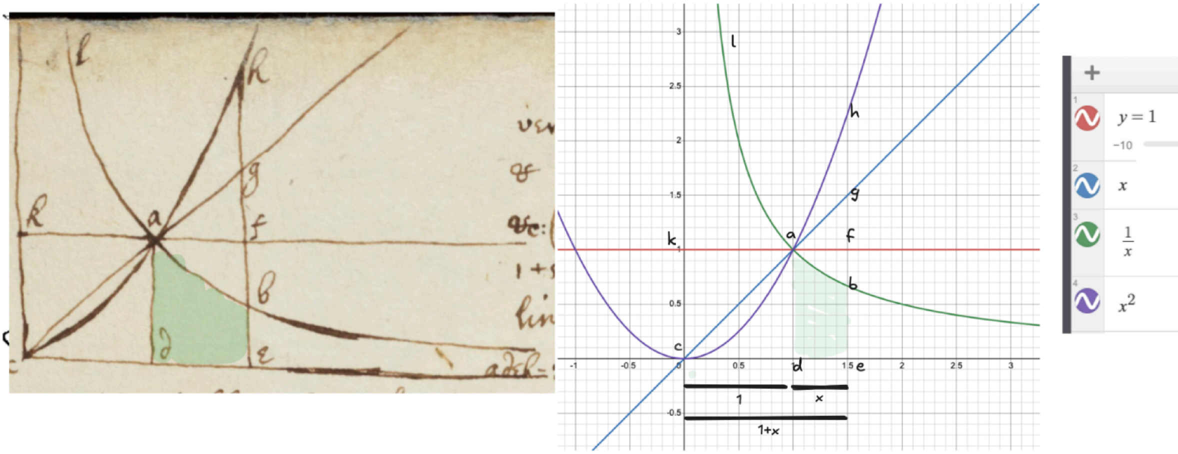


Figure 1: Sketch to show the curves Newton is considering to find the integral of $\frac{1}{x'}$ from 1 to 1+x, on the right is a facelift done with Desmos

We use x' and not x because x is reserved for the distance between the points d ($x' = 1$) and e ($x' = 1 + x$). This may seem strange but is necessary for Pascal's triangle coefficients to appear in the integrals that follow.

Very confusing notation. Probably better would be for example X (capital X).

[1]https://uu.brightspace.com/d2l/le/lessons/44275/topics/498036, page 191

[2]For one, Newton manipulates infinite series as if they were finite, without first checking that the series converges

Not strange because we want to avoid integrating from the vertical asymptote of the curve. Therefore starting the integration one step in is natural.

Then, shall the areas of those lines proceed in this progression. $* = adeb$, $x = adef$, $x + \frac{xx}{2} = adeg$, $adeh = x + \frac{2xx}{2} + \frac{x^3}{3}$, $x + \frac{3xx}{2} + \frac{3x^3}{3} + \frac{x^4}{4}$, $x + \frac{4xx}{2} + \frac{6x^3}{3} + \frac{4x^4}{4} + \frac{x^5}{5}$ etc.

In other words, we calculate the following integrals:

$$\int_1^{1+x} \frac{1}{x'}\,\mathrm{d}x' = \star$$

$$\int_1^{1+x} \mathrm{d}x' = x\ \Big|_1^{1+x} = (1+x) - 1 = \mathbf{1}x$$

$$\int_1^{1+x} x'\,\mathrm{d}x' = \frac{x'^2}{2}\ \Big|_1^{1+x} = \frac{(1+x)^2-1}{2} = \mathbf{1}x + \mathbf{1}\frac{x^2}{2}$$

$$\int_1^{1+x} x'^2\,\mathrm{d}x' = \frac{x'^3}{3}\ \Big|_1^{1+x} = \frac{(1+x)^3-1}{3} = \mathbf{1}x + \mathbf{2}\frac{x^2}{2} + \mathbf{1}\frac{x^3}{3}$$

$$\int_1^{1+x} x'^3\,\mathrm{d}x' = \frac{x'^4}{4}\Big|_1^{1+x} = \frac{(1+x)^4-1}{4} = \mathbf{1}x + \mathbf{3}\frac{x^2}{2} + \mathbf{3}\frac{x^3}{3} + \mathbf{1}\frac{x^4}{4}$$

$$\int_1^{1+x} x'^4 dx' = \frac{x'^5}{5}\ \Big|_1^{1+x} = \frac{(1+x)^5-1}{5} = \mathbf{1}x + \mathbf{4}\frac{x^2}{2} + \mathbf{6}\frac{x^3}{3} + \mathbf{4}\frac{x^4}{4} + \mathbf{1}\frac{x^5}{5}$$

$$\int_1^{1+x} x'^5 dx' = \frac{x'^5}{5} \mid_1^{1+x} = \frac{(1+x)^6 - 1}{6} = \mathbf{1}x + \mathbf{5}\frac{x^2}{2} + \mathbf{10}\frac{x^3}{3} + \mathbf{10}\frac{x^4}{4} + \mathbf{5}\frac{x^5}{5} + \mathbf{1}\frac{x^6}{6}$$

We have emphasized the coefficients of Pascal's triangle in **bold**. Newton then draws a table of the coefficients of the integrals he has, which allows him to extrapolate the pattern to establish his best guess for ⋆:

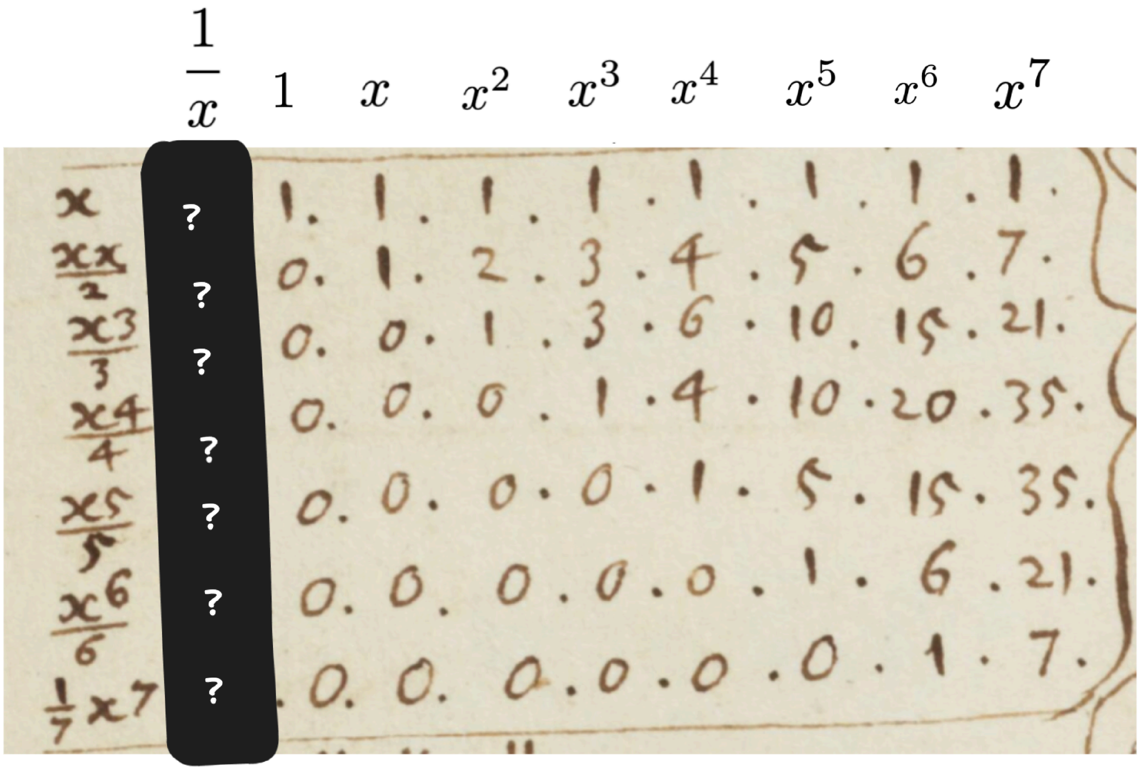
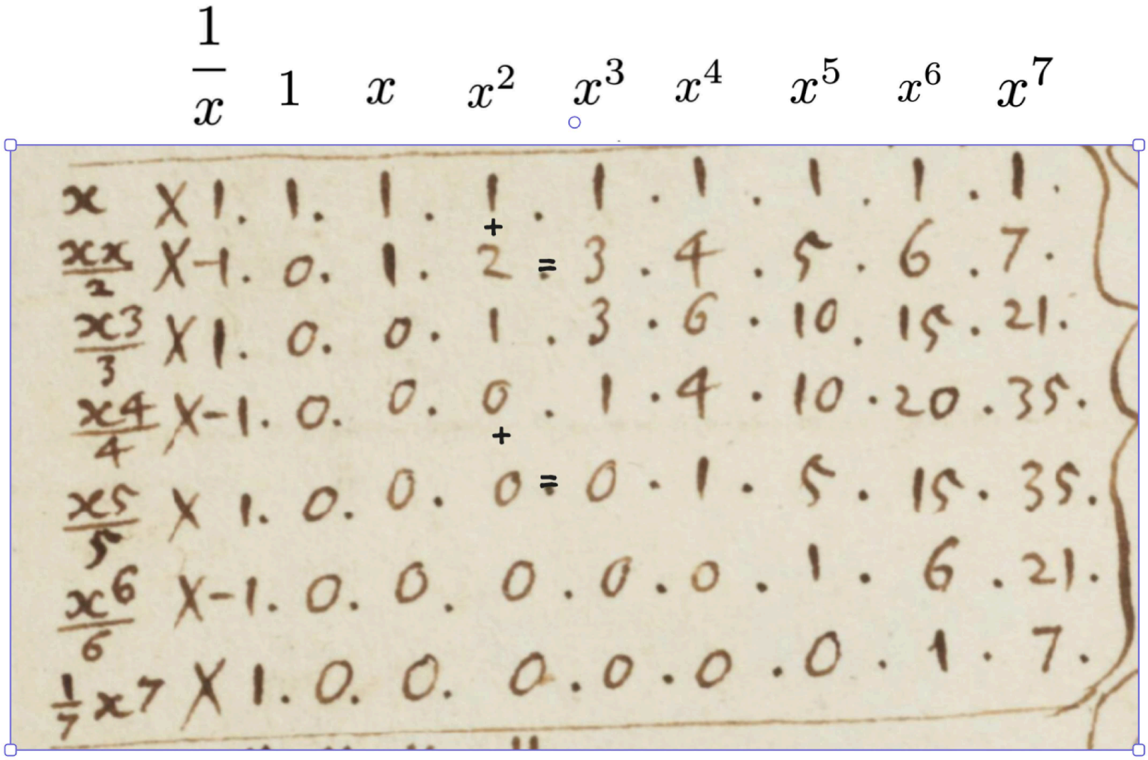


Figure 2: Each column represents the integral of a polynomial, where the coefficients are the same as the ones above in **bold**. Rows are increasingly higher terms in the expansions of the integrals. The integral of $\frac{1}{x}$ is the punchline of the 'proof' and we have redacted it at this stage. For the unredacted

version see Figure 3.

As in this table. In which the first area is also inserted.



	$\frac{1}{x}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
x ×	1.	1.	1.	1.	1.	1.	1.	1.	1.
$\frac{xx}{2}$ ×	−1.	0.	1.	2	3.	4.	5.	6.	7.
$\frac{x3}{3}$ ×	1.	0.	0.	1.	3.	6.	10.	15.	21.
$\frac{x4}{4}$ ×	−1.	0.	0.	0.	1.	4.	10.	20.	35.
$\frac{x5}{5}$ ×	1.	0.	0.	0	0.	1.	5.	15.	35.
$\frac{x6}{6}$ ×	−1.	0.	0.	0.	0.	0.	1.	6.	21.
$\frac{1}{7}x7$ ×	1.	0.	0.	0.	0.	0.	0.	1.	7.

Figure 3: Same as Figure 2 but with the coefficients for the integral of $\frac{1}{x}$ included. I have added some + and = signs to illustrate Newton's statement that *"The sum of any figure and the figure above it is equal to the figure next to it"*

The composition of which table may be

deduced from hence; viz: The sum of any figure and the figure above it is equal to the figure following it.

By writing the coefficients in a table and viewing Pascal's triangle as a square, this allows us to infer a 'missing' sequence which in this case coincides with the integral of 1/x.

Newton doesn't explicitly state his reasoning for the coefficients of the integral of $\frac{1}{x}$, but we can guess the argument goes something like this: First, the top-left element is probably 1, because all the topmost elements are 1. Second, work out all the subsequent numbers by the rule that "The sum of any figure and the figure above it is equal to the figure following it". This leads Newton to his punchline:

By which table it may appear that the area of the Hyperbola *adeb* is $x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10}$ etc.

Newton is guarded in his conclusions. "May appear" is not the same as "has been shown".[3] And indeed, all he has done is provide some numerical evidence that *if* there is a power series expansion for $\frac{1}{x}$, then this particular expansion would respect the pattern.

[3]This observation was made by Niccolò Guicciardini, Università degli Studi di Milano in his video exposition of this manuscript, available at: https://cudl.lib.cam.ac.uk/view/MS-ADD-03958/139

I suspect that Newton's "may" is synonymous with "can". "One can now conclude..." You can test this hypothesis by searching for other occurrences of the word "may" in Newton's writings. Do they express doubt or do they simply mean "can"?

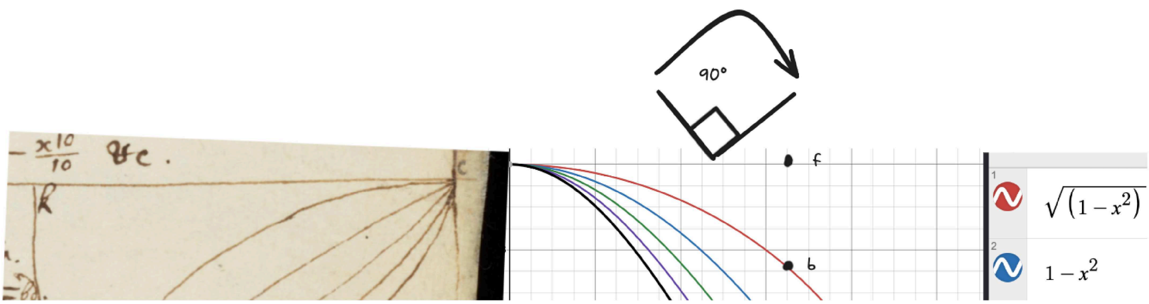
Next, Newton turns his attention to integrating a half-circle between arbitrary points on the x-axis.

Integral of Circle

While Wallis[4] had expressed π as a ~~power series,~~ infinite product Newton generalized his result by finding the integral of a semi-circle between any two points on the x-axis.

> Suppose that *adck* is a Square, *abc* a circle, *age* a Parabola, etc. and that $de = x$ and *ad* is parallel to $fe = 1 = bc$. And that the progression in which the lines *fe*, *be*, *ge*, *he*, *ie*, *ne*, etc. proceeds is 1, $\sqrt{1-xx}$, $1-xx$, $\overline{1-xx}\sqrt{1-xx}$, $1-2xx+x^4$, $\overline{1-2xx+x^4}\sqrt{1-xx}$, $1-3xx+3x^4-x^6$, etc.

Like with the hyperbola, Newton begins his study of the area of a circle by drawing an array of curves with the form $(1-x^2)^m$, for both integer and half-integer m.



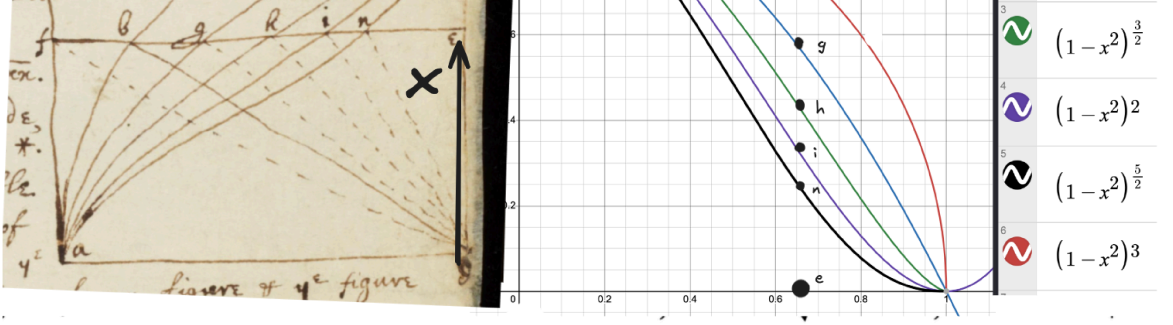


Figure 4: Left: Newton's original sketch. Right: We rotated Newton's sketch 90° clockwise to make sense of it. In modern notation, he is looking for patterns in the integrals of the form $\int_0^x (1-x^2)^m dx$ for $m = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ etc.

Newton then proceeds to integrate the easy ones:

> Then will their areas *fade*, *bade*, *gade*, *hade*, *iade*, etc. be in this progression $x, *, x - \frac{xxx}{3}, *, x - \frac{2}{3}x^3 + \frac{1}{5}x^5, *, x - \frac{3x^3}{3} + \frac{3x^5}{5} - \frac{x^7}{7}, *, x - \frac{4x^3}{3} + \frac{6x^5}{5} - \frac{4x^7}{7} + \frac{x^9}{9}$, etc:

In other words, he has just done these integrals:

[4]As claimed in Strogatz, https://uu.brightspace.com/d2l/le/lessons/44275/topics/498037

$$\int_0^x (1-x^2)^0 \, \mathrm{d}x = \mathbf{1}x$$

$$\int_0^x (1-x^2)^{\frac{1}{2}} \, \mathrm{d}x = \star$$

$$\int_0^x (1-x^2) \, \mathrm{d}x = \mathbf{1}x + \mathbf{1}\left(-\frac{x^3}{3}\right)$$

$$\int_0^x (1-x^2)^{\frac{3}{2}} \, \mathrm{d}x = \star$$

$$\int_0^x (1-x^2)^2 \, \mathrm{d}x = \int_0^x (1-2x^2+x^4) \, \mathrm{d}x = \mathbf{1}x + \mathbf{2}\left(-\frac{x^3}{3}\right) + \mathbf{1}\frac{x^5}{5}$$

$$\int_0^x (1-x^2)^{\frac{5}{2}} \, \mathrm{d}x = \star$$

$$\int_0^x (1-x^2)^3 \, \mathrm{d}x = \int_0^x (1-3x^2+3x^4-x^6) \, \mathrm{d}x = \mathbf{1}x + \mathbf{3}\left(-\frac{x^3}{3}\right) + \mathbf{3}\frac{x^5}{5} + \mathbf{1}\left(-\frac{x^7}{7}\right)$$

$$\int_0^x (1-x^2)^{\frac{7}{2}} \, \mathrm{d}x = \star$$

$$\int_0^x (1-x^2)^4 \, \mathrm{d}x = \int_0^x (1-4x^2+6x^4-4x^6+x^8) \, \mathrm{d}x = \mathbf{1}x + \mathbf{4}\left(-\frac{x^3}{3}\right) + \mathbf{6}\frac{x^5}{5} + \mathbf{4}\left(-\frac{x^7}{7}\right) + \mathbf{1}\frac{x^9}{9}$$

Again, we have arranged things such that Pascal’s triangle coefficients are now in **bold**. Note also the alternating signs between terms.

> as in this table following in which the indeterminate terms are inserted.

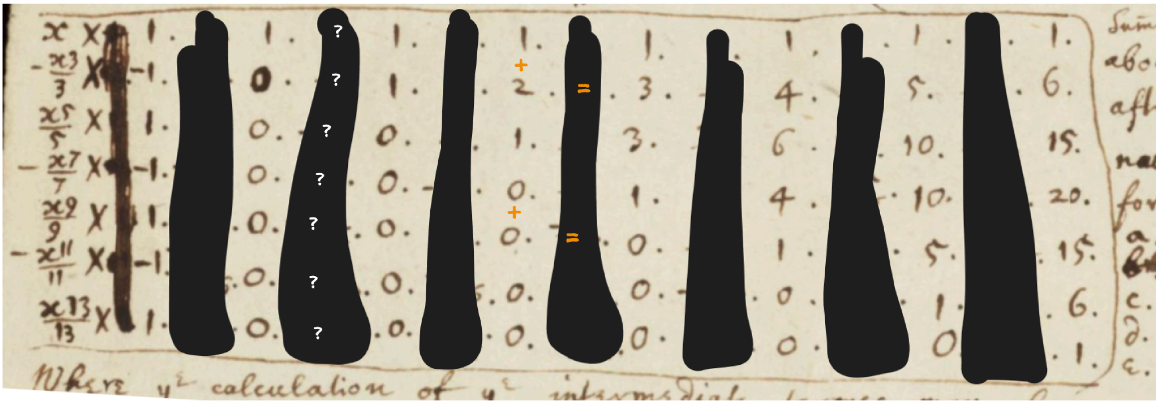


Figure 5: Each column represents the integral of a polynomial of the form $(1-x^2)^m$ for $m = -1, \frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$ except we have redacted the values $m = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ etc. The fourth column (redacted) contains the case of the circle $m = \frac{1}{2}$. Newton does not draw the graphs for $m = -1, -\frac{1}{2}$ but he does does treat them in this table. Each row represents a term in the series expansion for the aforementioned column.

Like in his treatment of the hyperbola, Newton spots Pascal's triangle:

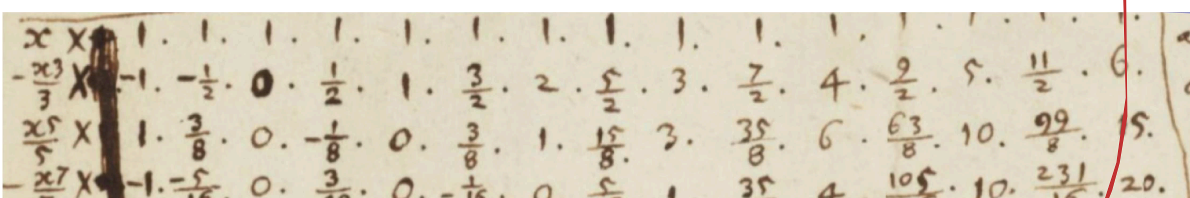
The property of which table is that the sum of any figure and the figure above it is equal to the figure next after it save one.

Newton then creates the following small finite difference table[5]

Also the numeral progressions are of these forms.

	0	1	2	3	...	n
0	a	a	a	a		a
1	b	$a+b$	$2a+b$	$3a+b$	...	$na+b$
2	c	$b+c$	$a+2b+c$	$3a+3b+c$		⋮
3	d	$c+d$	$b+2c+d$	$a+3b+3c+d$		
4	e	$d+e$	$c+2d+e$	$b+3c+3d+e$		
⋮						
m						

Where the calculation of the intermediate terms may be easily performed.



x X	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$-\frac{x^3}{3}$ X	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6
$\frac{x^5}{5}$ X	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15
$-\frac{x^7}{7}$ X	−1	$-\frac{5}{16}$	0	3/[illegible]	0	$-\frac{1}{16}$	0	5/[illegible]	1	35/[illegible]	4	105/[illegible]	10	$\frac{231}{16}$	20

I would think that we would seek to find such patterns (in modern terms, expressions in terms of n and m) and then infer for fractional entries which is equivalent to using special cases of the binomial formula.

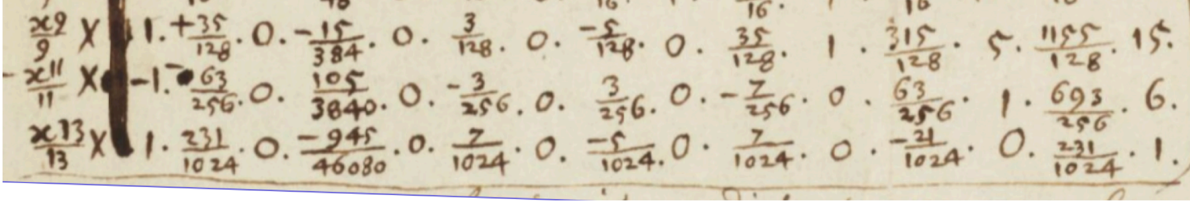


Figure 6: The unredacted table which includes the intermediate terms

We do not see how the calculation of the intermediate terms 'may be easily performed' from the finite difference table unless we take into account Newton's second letter to Leibniz, in which he explains that he obtained a general formula for the binomial coefficients for integer m, and then plugged in $m = \frac{1}{2}$ to get the value for the circle.

The logic goes something like this: First, the coefficient of x in the expansion of $\sqrt{1-x^2}$ must be 1, since the first row consists entirely of 1s. So $a = 1$. The coefficient of $-\frac{x^3}{3}$ must be $\frac{1}{2}$ because the second row is an arithmetic series $-1, 0, 1, 2, 3, 4$. So $b = \frac{1}{2}$. Next, we spot the following formula for generating successive terms of Pascal's triangle. If the second row is m, then the third term is $m \times \frac{m-1}{2}$, the fourth term is $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, the fifth term is $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ and so on. For a fuller explanation of this please see our section on Newton's Epistola Posterior.

This is what what we would today recognize as the binomial coefficient, i.e the statement that the k^{th} term of the binomial expansion of $\left(1-x^2\right)^m$ is: $\frac{m!}{(m-k)!k!}(-1)^k \frac{x^{2k+1}}{2k+1}$.

[5] We follow Stedall in calling this a 'finite difference table'; although we do not quite understand how this table leads him to conclude the form of the binomial coefficients. We do see that substituting a=1 gives us back the coefficients of Pascal's triangle.

Root Extraction: A Series from Newton's 'De analysi'[6]

From Newton, 'De analysi', 1669, as published in *Analysis per quantitatum series fluxiones, ac differentias*, 1711, 6-7.

In the following extract, Newton shows another way of obtaining infinite series - by adapting an algorithm that was commonly used at the time for getting the square roots of numbers.

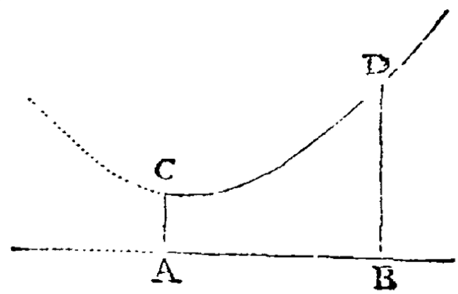
Exempla Radicem Extrahendo.

Si sit $\sqrt{aa+xx}=y$, Radicem sic extraho,

$$aa+xx\ \left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\ \text{\&c.}\right.$$

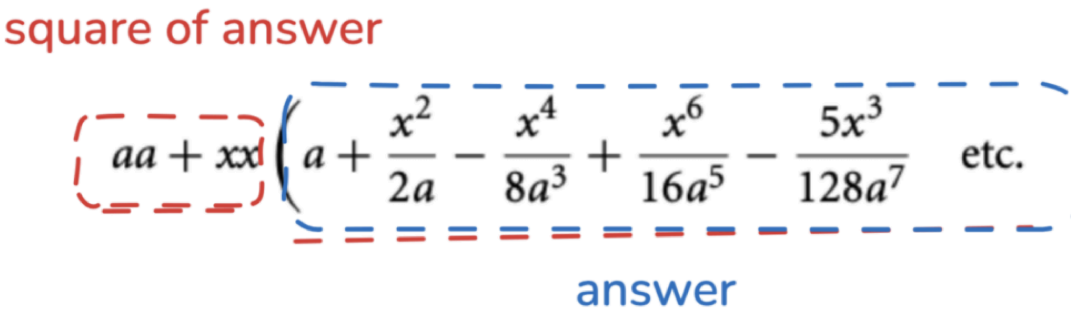
$$\begin{array}{l}
aa \\
\hline
0+xx \\
\quad xx+\dfrac{x^4}{4a^2} \\
\hline
\quad 0-\dfrac{x^4}{4a^2} \\
\qquad -\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6} \\
\hline
\qquad\quad 0+\dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6} \\
\qquad\qquad +\dfrac{x^6}{8a^4}+\dfrac{x^8}{16a^6}-\dfrac{x^{10}}{64a^8}+\dfrac{x^{12}}{256a^{10}} \\
\hline
\qquad\qquad\quad 0-\dfrac{5x^8}{64a^6}+\dfrac{x^{10}}{64a^8}-\dfrac{x^{12}}{256a^{10}}
\end{array}$$

&c.



Unde,

The first thing to note is that the left bracket has nothing to do with bracketing. Instead, it is supposed to separate the square of our answer from our answer:



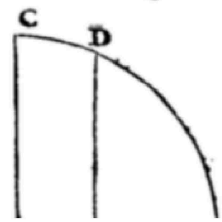
The algorithm goes as follows: We start with guessing a number whose square will equal a^2. This is obviously a. Then we take our guess (a), square it (a^2), and take it away from our target answer $a^2 + x^2$. The remainder is x^2. We take the remainder, divide it by $2a$, and add this value to our old guess (a), to produce our new guess ($a + \frac{x^2}{2a}$). Then repeat the process by squaring our new guess ($a^2 + x^2 + \frac{x^4}{4a^2}$) and taking this away from our target (still $a^2 + x^2$). The remainder ($-\frac{x^4}{4a^2}$) is, like before,

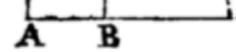
[6] from Newton, 'De analysi', 1669, as published in Analysis per quantitatum series, fluxiones, ac differentias, 1711, 6–7, available on https://uu.brightspace.com/d2l/le/lessons/44275/topic, page 194

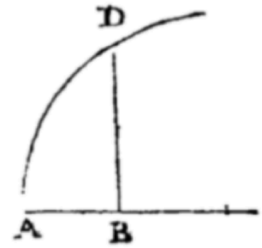
divided by 2a and added to our old guess ($a+\frac{x^2}{2a}$), creating a new guess ($a+\frac{x^2}{2a}-\frac{x^4}{8a^3}$). This process can continue forever.

In the next step Newton takes this polynomial expansion of the curve, and integrates it term by term to get the area ABCD.

> PER ÆQUATIONES INFINITAS. 7
>
> Unde, pro Æquatione $\sqrt{aa+xx}=y$, nova producitur, viz. $y = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Et (per Reg. 2.) Area quæsita ABDC erit $= ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. Et hæc est Quadratura Hyperbolæ.







“Et haec est Quadratura Hyperbola” - thus we have squared the hyperbola.

In the following line-by-line commentary, we will use screenshots from the translation given in The History of Mathematics: A Source-Based Approach (2022), Volume 2, chapter 4.2.

Newton's Epistola Prior

Though the modesty of Mr Leibniz, in the extracts from his letter which you have lately sent me, pays great tribute to our countrymen for a certain theory of infinite series, about which there now begins to be some talk, yet I have no doubt that he has discovered not only a method for reducing any quantities whatever to such series, as he asserts, but also various shortened forms, perhaps like our own, if not even better. Since, however, he very much wants to know what has been discovered in this subject by the English, and since I myself fell upon this theory some years ago, I have sent you some of those things which occurred to me in order to satisfy his wishes, at any rate in part.

This is the introduction of the first letter Newton sent to Leibniz (sent in 1676 via Oldenburg) about the infinite series he found. In this letter Newton is mostly just showing off the series he found and how it can be used. This introduction is not very relevant for the mathematics, so I will comment on it no further.

Fractions are reduced to infinite series by division; and quantities by extraction of the roots, by carrying out those operations in the symbols just as they are commonly carried out in decimal numbers. These are the foundations of these reductions: but extractions of roots are much shortened by this theorem,

$$(P+PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + etc.$$

where $P+PQ$ signifies the quantity whose root or even any power, or the root of a power, is to be found; P signifies the first term of

"Fractions are reduced to infinite series by division; and quantities [are reduced] by extraction of the roots, by carrying out the operations just as they are normally carried out for decimal numbers. These are the foundations of these reductions: and this theorem also shortens the extractions of roots,

$$(P+PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + etc$$

where we want to find the root or any (root of the) power of $P+PQ$;"

You are repeating Newton's words with minimal changes for some reason. Is this meant as a commentary or explanation? That makes little sense since you are just writing the same thing twice.

A commentary could instead better explain how Newton's series corresponds to the modern binomial series, and what the significance or advantages or disadvantages are of the differences between his approach and notation compared to a modern approach.

or the root of a power, is to be found; P signifies the first term of that quantity, Q the remaining terms divided by the first, and m/n the numerical index of the power of $P + PQ$, whether that power is integral or (so to speak) fractional, whether positive or negative. For

“P is the first term of the quantity, Q is the remaining terms divided by P, and m/n is the numerical index of the power of $P + PQ$, whether that power is an integer or fractional, positive or negative.”

tegral or (so to speak) fractional, whether positive or negative. For
as analysts, instead of aa, aaa, etc., are accustomed to write a^2, a^3, etc., so instead of $\sqrt{a}, \sqrt{a^3}, \sqrt{c : a^5}$, etc.[8] I write $a^{1/2}, a^{3/2}, a^{5/3}$, and instead of $1/a, 1/aa, 1/a^3$, I write a^{-1}, a^{-2}, a^{-3}. And so for $\frac{aa}{\sqrt{c:(a^3+bbx)}}$ I write $aa(a^3 + bbx)^{-1/3}$ and for

$$\frac{aab}{\sqrt{c : \{(a^3 + bbx)(a^3 + bbx)\}}}$$

I write $aab(a^3 + bbx)^{-2/3}$: in which last case, if $(a^3 + bbx)^{-2/3}$ is supposed

Here Newton explains his notation.

I write $aab(a^3 + bbx)^{-2/3}$: in which last case, if $(a^3 + bbx)^{-2/3}$ is supposed
to be $(P + PQ)^{m/n}$ in the Rule, then P will be equal to a^3, Q to bbx/a^3

to be $(P + PQ)^{m/n}$ in the Rule, then P will be equal to a^3, Q to bbx/a^3,
m to -2, and n to 3. Finally, for the terms found in the quotient in the

"if $(a^3 + b^2x)^{-\frac{2}{3}}$ is supposed to be $(P + PQ)^{\frac{m}{n}}$, in the function, then $P = a^3, Q = \frac{b^2x}{a^3}, m = -2$ and $n = 3$."

m to -2, and n to 3. Finally, for the terms found in the quotient in the course of the working I employ A, B, C, D, etc., namely, A for the first term, $P^{m/n}$; B for the second term, m/nAQ; and so on. For the rest, the use of the rule will appear from the examples.

"Finally, for the coefficients used in the function, I use A, B, C, D, etc., namely, A to refer to the first coefficient, $P^{m/n}$; B for the second coefficient, $\frac{m}{n}AQ$; and so on. For the rest, the use of the rule will appear from the examples."

Here Newton explains how the recursive part of the function works/is used.

Example 1

$$\sqrt{(c^2+x^2)} \text{ or } (c^2+x^2)^{1/2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256[c]^9} + \ etc.$$

For in this case $P = c^2, Q = x^2/c^2, m = 1, n = 2,\ A\,(= P^{m/n} = (cc)^{1/2}) = c$, $B\,(= (m/n)AQ) = x^2/2c$, $C\left(= \frac{m-n}{2n}BQ\right) = -\frac{x^4}{8c^3}$; and so on.

"Example 1

$\sqrt{c^2+x^2} = (c^2+x^2)^{1/2} = c + \frac{x^2}{2c} + \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256c^9} +$ etc. For this case, $P = c^2, Q = \frac{x^2}{c^2}, m = 1, n = 2, A\left(= P^{\frac{m}{n}}\right) = c, B\ \left(= \frac{m}{n}AQ\right) = \frac{x^2}{2c}, C\left(= \frac{m-n}{2n}BQ\right) = -x^4/8c^3$; and so on."

Example 2

$$\sqrt[5]{(c^5+c^4x-x^5)},$$

i.e.

$$(c^5+c^4x-x^5)^{1/5} = c + \frac{c^4x-x^5}{5c^4}[+]\frac{-2c^8x^2+4c^4x^6-2x^{10}}{25c^9} + etc.$$

as will be evident on substituting 1 for m, 5 for n, c^5 for P and $(c^4x-x)^5/c^5$ for Q, in the rule quoted above. Also $-x^5$ can be substituted for P and $(c^4x+c^5)/(-x^5)$ for Q. The result will then be

$$\sqrt[5]{(c^5+c^4x-x^5)} = -x + \frac{c^4x+c^5}{5x^4} + \frac{2c^8x^2+4c^9x+[2]c^{10}}{25x^9} + etc.$$

“Example 2

$\sqrt[5]{c^5+c^4x-x^5} = (c^5+c^4x-x^5)^{1/5} = c + \frac{c^{4x}-x^5}{5x^4} + \frac{2c^8x^2+4c^9x+2c^{10}}{25x^9}$ +etc. which becomes clear by substituting $m=1, n=5, P=c^5$ and $Q=\frac{c^4x-x^5}{c^5}$, in the function stated at the beginning. You can also do the following substitution $P=-x^5$ and $Q=-\frac{c^4x+c^5}{x^5}$. The result will then be $\sqrt[5]{c^5+c^4x-x^5} = -x + \frac{c^4x+c^5}{5x^4} + \frac{2c^8x^2+4c^9x+2c^{10}}{25x^9}$ +etc. ”

The first method is to be chosen if x is very small, the second if it is very large. → why?

As mentioned at the beginning of the line-by-line review of this letter, Newton is talking about the series he found but not explaining how he found it or even convinced himself that it works. That happens in the second letter.

Newton's Epistola Posterior

I can hardly tell with what pleasure I have read the letters of those very distinguished men Leibniz and Tschirnhaus. Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else. But what he has scattered elsewhere throughout his letter is most worthy of his reputation — it leads us also to hope for very great things from him. The variety of ways by which the same goal is approached has given me the greater pleasure, because three methods of arriving at series of that kind had already become known to me, so that I could scarcely expect a new one to be communicated to us. One of mine I have described before; I now add another, namely, that by which I first chanced on these series — for I chanced on them before I knew the divisions and extractions of roots which I now use. And an explanation of this will serve to lay bare, what Leibniz desires from me, the basis of the theorem set forth near the beginning of the former letter.

This is the introduction of the second letter Newton sent to Leibniz (via Oldenburg) about the infinite series he had found. In this letter, Newton appears much more cordial and explains how he found the series and convinced himself that this series actually works the way he thinks it does. Note that in this letter he does not give any proof, he just explains how he found it. Not much mathematics is happening in this section, so I will not comment on it further.

At the beginning of my mathematical studies, when I had met with the works of our celebrated Wallis, on considering the series by the intercalation of which he himself exhibits the area of the circle and the hyperbola, the fact that, in the series of curves whose common base or axis is x and the ordinates

$$(1-x^2)^{0/2}, (1-x^2)^{1/2}, (1-x^2)^{2/2}, (1-x^2)^{3/2}, (1-x^2)^{4/2}, (1-x^2)^{5/2}, etc,$$

if the areas of every other of them, namely

$$x^3, x-\frac{1}{3}x^3, x-\frac{2}{3}x^3+\frac{1}{5}x^5, x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7, etc.$$

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1-x^2)^{1/2}$ is the circle: in order to interpolate

“At the beginning of my mathematical studies, when I studied the works of the celebrated Wallis on the infinite series expansions he derived and used to compute the area of the circle and hyperbola, I noticed that, for the series of functions of the form $y = (1-x)^{\frac{n}{2}}$, with $n \in \mathbb{N}_0$, and all sharing the same x-axis, if the area of every other function could be determined, we should have the areas of the intermediate functions, of which the first $(1-x)^{\frac{1}{2}}$ is the circle”

Thus, what Newton is saying here is that he got inspired by the work of Wallis to look at a set of functions, for the area was known if n is even. He wanted to see if he could find a pattern for the integrals for odd n.

ones, of which the first $(1-x^2)^{1/2}$ is the circle: in order to interpolate these series I noted that in all of them the first term was x and the second terms $\frac{0}{3}x^3$, $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, etc., were in arithmetical progression, and hence that the first two terms of the series to be intercalated ought to be $x - \frac{1}{3}(\frac{1}{2}x^3)$, $x - \frac{1}{3}(\frac{3}{2}x^3)$, $x - \frac{1}{3}(\frac{5}{2}x^3)$, etc. To intercalate

“in order to interpolate these series in an arithmetic progression, I observed that in all of them the first term was x and the second terms $\frac{0}{3}x^3, \frac{1}{3}x^3, \frac{2}{3}x^3, \frac{3}{3}x^3$, etc., were an arithmetic sequence, so the first to terms of the series to be deduced would have to be $x -$

$\frac{1}{3}\left(\frac{1}{2}x^3\right), x-\frac{1}{3}\left(\frac{3}{2}x^3\right), x-\frac{1}{3}\left(\frac{5}{2}x^3\right)$, etc."

Here Newton is describing how he deduced the first two terms for the series expansion of the area of $(1-x)^{\frac{n}{2}}, n \in \mathbb{N}$.

lated ought to be $x-\frac{1}{3}(\frac{1}{2}x^3)$, $x-\frac{1}{3}(\frac{3}{2}x^3)$, $x-\frac{1}{3}(\frac{5}{2}x^3)$, etc. To intercalate the rest I began to reflect that the denominators $1, 3, 5, 7$, etc., were in arithmetical progression, so that the numerical coefficients of the numerators only were still in need of investigation. But in the alter-

"To find the rest, I noticed that the denominators 1, 3, 5, 7, etc., were an arithmetic sequence, thus now only the numerical coefficients of the numerators were unknown."

numerators only were still in need of investigation. But in the alter-
nately given areas these were the figures of powers of the number 11, namely of these, $11^0, 11^1, 11^2, 11^3, 11^4$, that is, first 1; then $1, 1$; thirdly, $1, 2, 1$; fourthly $1, 3, 3, 1$; fifthly $1, 4, 6, 4, 1$, etc. And so I began to inquire

"But in the areas of the functions we already knew, these [numerators] were the digits appearing in the powers of the number 11, namely the digits of these, $11^0, 11^1, 11^2, 11^3, 11^4$, that is, first 1; then 1, 1; thirdly 1, 2, 1; fourthly 1, 3, 3, 1; fifthly 1, 4, 6, 4, 1, etc."

This numerical pattern corresponds to what is now called Pascal's triangle (i.e., the binomial coefficients). ①

$1, 2, 1$; fourthly $1, 3, 3, 1$; fifthly $1, 4, 6, 4, 1$, etc. And so I began to inquire how the remaining figures in these series could be derived from the first two given figures, and I found that on putting m for the second figure, the rest would be produced by continual multiplication of the terms of this series,

$$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times etc.$$

"And so I began to wonder how the remaining digits in these series could be derived from the first two given digits, and I found that when replacing the second number with m, the rest of the digits would be produced by continues multiplication of the terms of this series, $\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ etc."

Newton here describes the recursive rule that generates the numbers for the numerators. He next gives an example which helps clarify:

For example, let $m = 4$, and $4 \times \frac{1}{2}(m-1)$, that is 6 will be the third term, and $6 \times \frac{1}{}(m-2)$, that is 4 the fourth, and $4 \times \frac{1}{}(m-3)$, that is 1

① This patten immidiately goes wrong in the very next step. why? This is the kind of thing an actual commentary on a curious reader could reflect on. Since you are just parroting Newton's own words you are not demonstrating any such reflection.

term, and $6 \times \frac{1}{3}(m-2)$, that is 4 the fourth, and $4 \times \frac{1}{4}(m-3)$, that is 1 the fifth, and $1 \times \frac{1}{5}(m-4)$, that is 0 the sixth, at which term in this case the series stops. Accordingly, I applied this rule for interposing series

"For example, let $m = 4$, then $4 \times \frac{1}{2}(m-1) = 6$ will be the numerator of the third term, and $6 \times \frac{1}{3}(m-2) = 4$ the numerator of the fourth, and $4 \times \frac{1}{4}(m-3) = 1$ the fifth, and $1 \times \frac{1}{5}(m-4) = 0$ the sixth, at which term in this case the series stops".

Here Newton gives an example to find the binomial coefficients for $(a+b)^4$, where he assumes that you know that the first coefficient is 1 (because for each case he discussed here, the series starts with $(1)x$.

the series stops. Accordingly, I applied this rule for interposing series among series, and since, for the circle, the second term was $\frac{1}{3}(\frac{1}{2}x^3)$, I put $m = \frac{1}{2}$, and the terms arising were

$$\frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \text{ or } -\frac{1}{8}, \quad -\frac{1}{8} \times \frac{\frac{1}{2}-2}{3} \text{ or } +\frac{1}{16}, \quad \frac{1}{16} \times \frac{\frac{1}{2}-3}{4} \text{ or } -\frac{5}{128},$$

and so to infinity. Whence I came to understand that the area of the

"Accordingly, I used this rule for inserting the new series between the previously known series, and seeing as for the circle the second term was $\frac{1}{3}\left(\frac{1}{2}x^3\right)$, I put $m = \frac{1}{2}$, and I found $\frac{1}{2} \times \frac{\frac{1}{2}-1}{2} = -\frac{1}{8}, -\frac{1}{8} \times \frac{\frac{1}{2}-2}{3} = \frac{1}{16}, \frac{1}{16} \times \frac{\frac{1}{2}-3}{4} = -\frac{5}{128}$, and so on to infinity."

After Newton saw that his formula worked for the known series, he simply applied the same rules to the series he was interested in and that got him this result.

and so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} \; etc.$$

And by the same reasoning the areas of the remaining curves, which were to be inserted, were likewise obtained: as also the area of the hyperbola and of the other alternate curves in this series

$$(1 - x^2)^{0/2},\ (1 - x^2)^{1/2},\ (1 - x^2)^{2/2},\ (1 - x^2)^{3/2},\ \text{etc.}$$

"This is how I came to understand that the area of the segment of the circle that I was interested in was $x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9}$ etc. And using the same reasoning, the areas of the remaining functions that were to be inserted, were found, as well as the area of the hyperbola and of the other functions in this series $(1-x)^{n/2}, n \in \mathbb{N}$."

And the same theory serves to intercalate other series, and that through intervals of two or more terms when they are absent at the same time. This was my first entry upon these studies, and it had certainly escaped my memory, had I not a few weeks ago cast my eye back on some notes.

"The same method applies to the interpolation of other series, even when one or several consecutive intermediate cases are missing. This was my first step into this field, and I would not have remembered it if I had not looked at my notes a few weeks ago."

But when I had learnt this, I immediately began to consider that the terms

$$(1-x^2)^{0/2},\ (1-x^2)^{2/2},\ (1-x^2)^{4/2},\ (1-x^2)^{6/2},\ etc.$$

that is to say,

$$1,\ 1-x^2,\ 1-2x^2+x^4,\ 1-3x^2+3x^4-x^6,\ etc.$$

could be interpolated in the same way as the areas generated by them: and that nothing else was required for this purpose but to omit the denominators $1, 3, 5, 7$, etc., which are in the terms expressing the areas;

"After I had learned this, I immediately considered that the series of functions $(1-x)^{2n/2}$, $n \in \mathbb{N}$ themselves could be expanded in the same way as the areas generated by them, and to that I only needed to omit the denominators 1, 3, 5, 7, etc., which are expressing the areas in the coefficients,"

this means that the coefficients of the terms of the quantity to be intercalated $(1-x^2)^{\frac{1}{2}}$, or $(1-x^2)^{\frac{3}{2}}$, or in general $(1-x^2)^m$, arise by the continued multiplication of the terms of this series continued multiplication of the terms of this series

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}\ etc.$$

"This means that the coefficients of the terms of the to be inserted function $(1-x)^{\frac{1}{2}}$, or $(1-x)^{\frac{3}{2}}$, or in general $(1-x)^m$, arise by the continued multiplication of the terms of this series $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ etc."

so that (for example)

$$(1-x^2)^{\frac{1}{2}} \text{ was the value of } 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6\ etc.,$$

$$\text{and } (1-x^2)^{\frac{3}{2}} \text{ was the value of } 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6\ etc.,$$

$$(1-x^2)^{\frac{1}{3}} \text{ was the value of } 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6\ etc.$$

Newton has generalized his formula so as to not only find the area below certain functions but also find a way to expand the functions themselves. Yes, this is the kind of thing a commentary should be explaining.

So then the general reduction of radicals into infinite series by that rule, which I laid down at the beginning of my earlier letter became known to me, and that before I was acquainted with the extraction of roots.

“Thus I came to understand the general method of expressing radicals as infinite series using the rule stated at the beginning of my earlier letter, and I discovered this before I had learned the procedure for extracting roots.” (1)

But once this was known, that other could not long remain hidden from me. For in order to test these processes, I multiplied

$$1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6, etc.$$

into itself; and it became $1-x^2$, the remaining terms vanishing by the continuation of the series to infinity. And even so $1-\frac{1}{3}x^2-\frac{1}{9}x^4-\frac{5}{81}x^6$, etc. multiplied twice into itself also produced $1-x^2$.

“Once I knew this, it was not long before I also understood that. In order to test this procedure, I multiplied $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6$, etc. with itself, and it became $1-x^2$, with the remaining terms vanishing due to the continuation of the series to infinity. Likewise, the

(1) The general before the particular: perhaps somewhat unexpected indeed. Something it might be interesting to reflect on. When does generalising a problem make it easier? why does it help in this case? Are there other such examples? Did Polya say something about it in How to solve it for example? These are examples of questions a curious reader might ask.

cube of $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6$, etc. also gave $1 - x^2$."

And as this was not only sure proof of these conclusions so too it guided me to try whether, conversely, these series, which it thus affirmed to be roots of the quantity $1 - x^2$, might not be extracted out of it in an arithmetical manner. And the matter turned out well. This was the form of the working in square roots.

$$
\begin{array}{llllll}
1 & -x^2(1 & -\frac{1}{2}x^2 & -\frac{1}{8}x^4 & -\frac{1}{16}x^6, \text{ etc.} \\
1 & & & & \\
\hline
0 & -x^2 & & & \\
 & -x^2 & +\frac{1}{4}x^4 & & \\
\hline
 & & -\frac{1}{4}x^4 & & \\
 & & -\frac{1}{4}x^4 & +\frac{1}{8}x^6 & +\frac{1}{64}x^8 \\
\hline
 & & 0 & -\frac{1}{8}x^6 & -\frac{1}{64}x^8
\end{array}
$$

"This was not only the proof of these series being correct, but it also led me to investigate whether these series, which were confirmed to be roots of $1 - x^2$, could be derived by an arithmetical procedure. And this worked. This was how I extracted the square root."

We explain the extraction of roots in our section on De Analysi (see page 10).

After getting this clear I have quite given up the interpolation of series, and have made use of these operations only, as giving more natural foundations. Nor was there any secret about reduction by division, an easier affair in any case.

"Having cleared this up, I have not done any more work on inserting intermediate cases between known series, and have only used the procedure, because it gives a more natural foundation. The reduction [of ~~roots~~] by division was no secret and easier to use."

no, not of roots

why?

The first part of the report is good.
The second part of the report just restates Newton's text with trivial changes in wording while demonstrating no understanding or thoughtful engagement with the text or critical reflection.