Newton & the General Binomial Theorem

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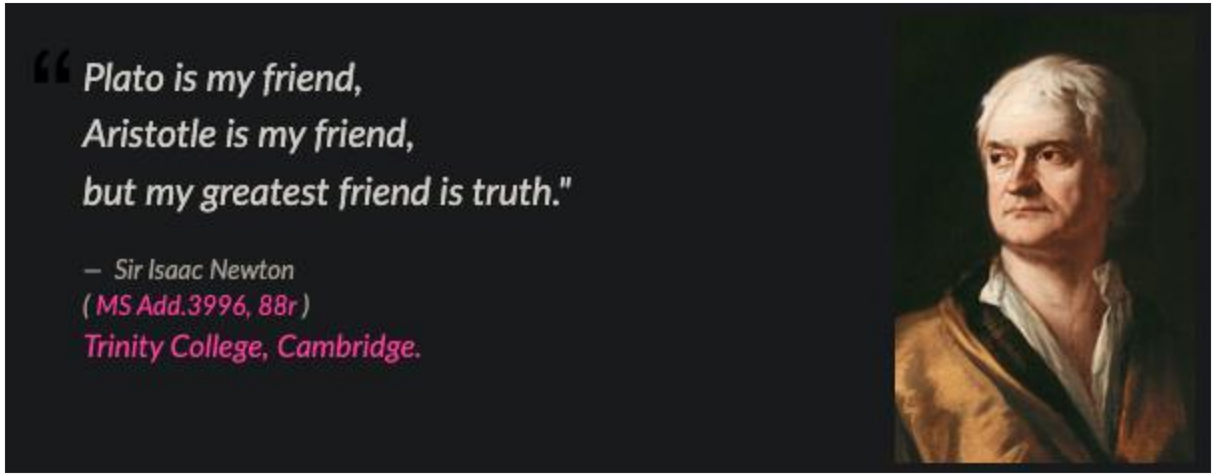
Seminar History of Mathematics 2026



Epistola Prior

Fractions are reduced to infinite series by division; and quantities by extraction of the roots, by carrying out those operations in the symbols just as they are commonly carried out in decimal numbers. These are the foundations of these reductions: but extractions of roots are much shortened by this theorem,

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + etc.$$

A for the first term, $P^{m/n}$; B for the second term, m/nAQ; and so on. For the rest, the use of the rule will appear from the examples.

Newton's *Epistola Prior*

Epistola Prior

- Modern notation of Generalized Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1}.$$

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Exempla Radicem Extrahendo.

Si sit $\sqrt{aa+xx}=y$, Radicem sic extraho,

$$aa+xx\ \left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\ \&c.\right.$$

$$\frac{aa}{0}+xx$$

$$\underline{xx+\frac{x^4}{4a^2}}$$

$$0-\frac{x^4}{4a^2}$$

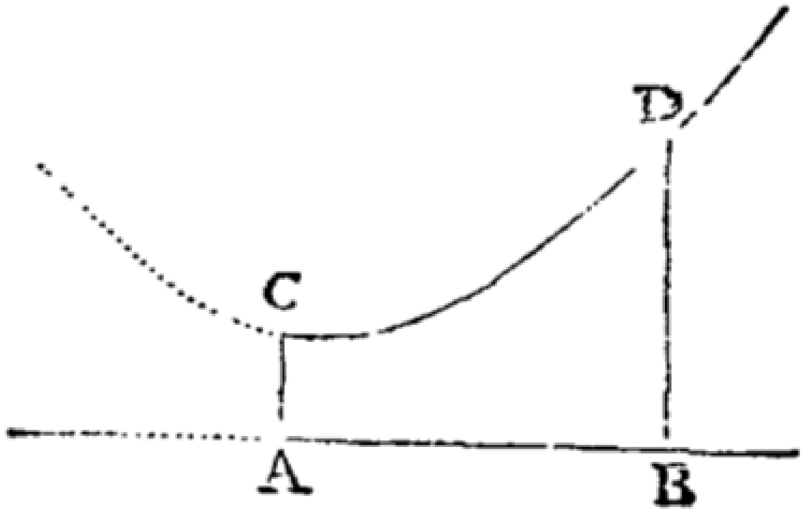
$$\underline{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}$$

$$0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\underline{+\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}}}$$

$$0-\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}}$$

&c.



Unde,

Newton, 1669, "Analysis by means of equations with an infinite number of terms" aka "De Analysi" (published 1711)

Epistola Posterior

At the beginning of my mathematical studies, when I had met with the works of our celebrated Wallis, on considering the series by the intercalation of which he himself exhibits the area of the circle and the hyperbola, the fact that, in the series of curves whose common base or axis is x and the ordinates

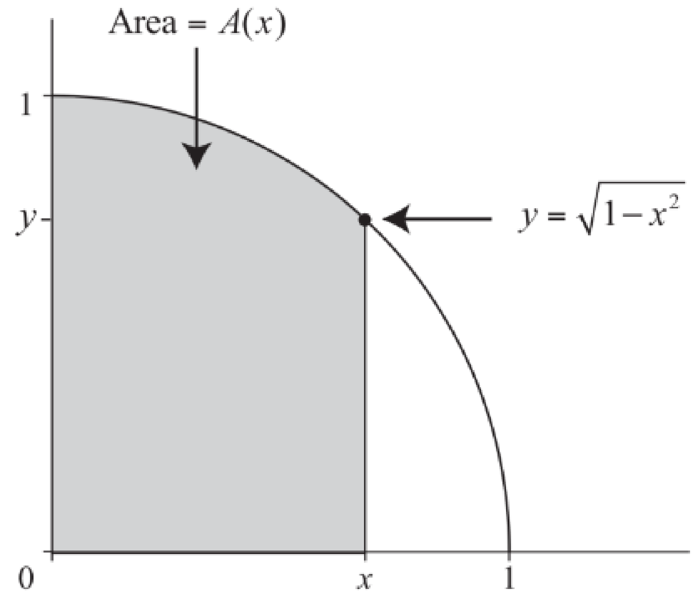
$$(1-x^2)^{0/2}, (1-x^2)^{1/2}, (1-x^2)^{2/2}, (1-x^2)^{3/2}, (1-x^2)^{4/2}, (1-x^2)^{5/2}, etc,$$

if the areas of every other of them, namely

$$x^3, x - \frac{1}{3}x^3, x - \frac{2}{3}x^3 + \frac{1}{5}x^5, x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7, etc.$$

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1-x^2)^{1/2}$ is the circle

Newton's *Epistola Posterior*



Strogatz

Epistola Posterior

Let's write A_n for the area under the curve $y = (1 - x^2)^{n/2}$, where $n = 0, 1, 2, \dots$. Then

$$A_0 = x$$
$$A_1 = ?$$
$$A_2 = x - \frac{1}{3}x^3$$
$$A_3 = ?$$
$$A_4 = x - \frac{2}{3}x^3 + \frac{1}{5}x^5$$
$$A_5 = ?$$
$$A_6 = x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7.$$

$$\frac{1}{3}\left(\frac{1}{2}x^3\right)$$
$$\frac{1}{3}\left(\frac{3}{2}x^3\right)$$
$$\frac{1}{3}\left(\frac{5}{2}x^3\right)$$

Denominators: 1, 3, 5, 7, etc.
Odd numbers in increasing order

Numerators: 1, 11, 121, 1331, 14641
Pascal's Triangle! Or as Newton said: powers of 11

Epistola Posterior

- Need to extend Pascal's triangle to halfway in between the rows...
- First, general formula for binomial coefficients of row *m:*

$$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}\text{, etc.}$$

- Next, just plug in ½:

[...] Accordingly I applied this rule for interposing series among series, and since, for the circle, the second term was $\frac{1}{3}\left(\frac{1}{2}x^3\right)$, I put $m = 1/2$, and the terms arising were

$$\frac{1}{2} \times \frac{\frac{1}{2}-1}{2} \text{ or } -\frac{1}{8}, \qquad -\frac{1}{8} \times \frac{\frac{1}{2}-2}{3} \text{ or } +\frac{1}{16}, \qquad \frac{1}{16} \times \frac{\frac{1}{2}-3}{4} \text{ or } -\frac{5}{128},$$

so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} \text{ etc.''}$$

Epistola Posterior

- Eureka! Interpolate curves in same way as area beneath the curve!
  - Omit denominators and reduce power by 1

this means that the coefficients of the terms of the quantity to be intercalated $(1-x^2)^{\frac{1}{2}}$, or $(1-x^2)^{\frac{3}{2}}$, or in general $(1-x^2)^m$, arise by the continued multiplication of the terms of this series

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \text{ etc.}$$

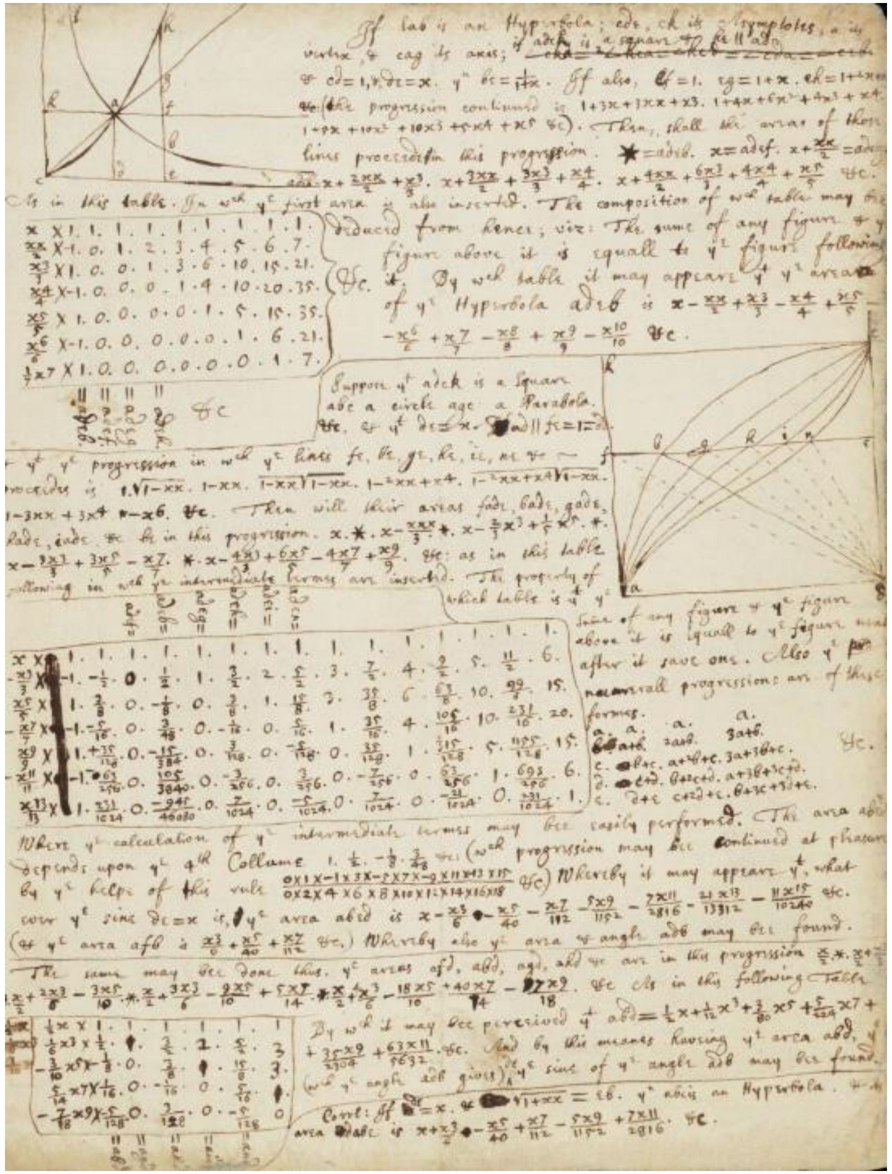
so that (for example)

$$(1-x^2)^{\frac{1}{2}} \text{ was the value of } 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 \text{ etc.,}$$

$$\text{and } (1-x^2)^{\frac{3}{2}} \text{ was the value of } 1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6 \text{ etc.,}$$

$$(1-x^2)^{\frac{1}{3}} \text{ was the value of } 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6 \text{ etc.}$$

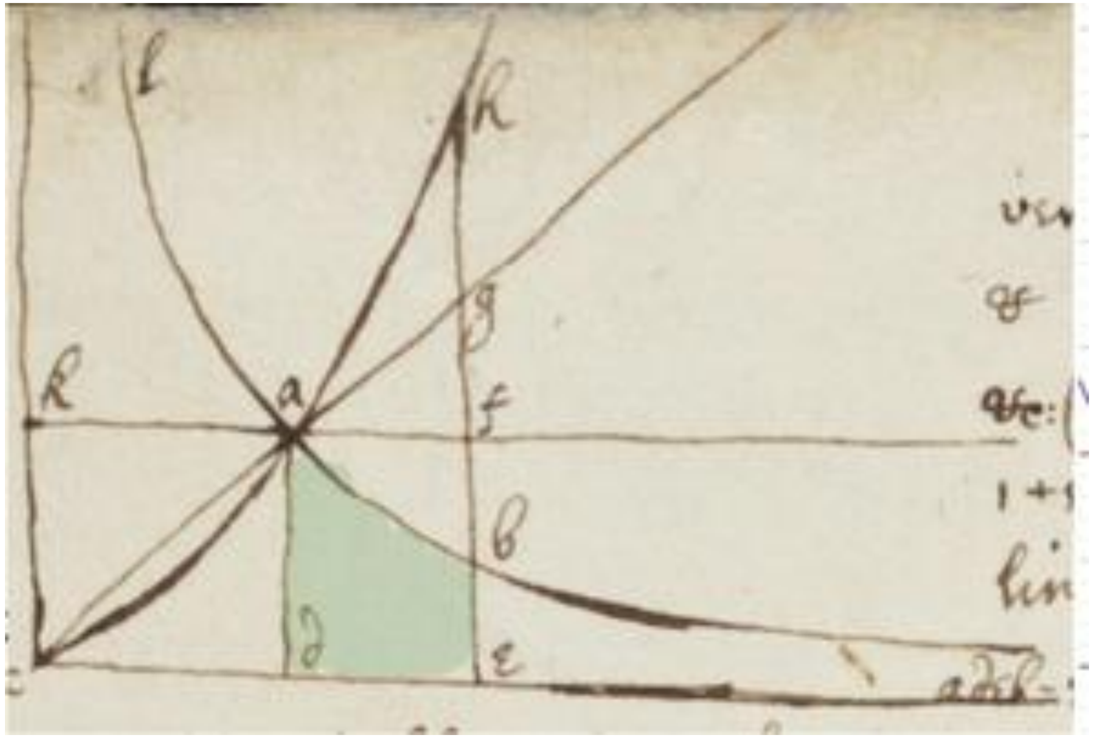
Early Papers



Extrapolating Pascal's triangle backwards

Finding the area of the hyperbola

Early Papers



$$\int_1^{1+x} \frac{1}{x'}\,\mathrm{d}x' = \star$$

$$\int_1^{1+x} \mathrm{d}x' = x\ |_1^{1+x} = (1+x) - 1 = \mathbf{1}x$$

$$\int_1^{1+x} x'\,\mathrm{d}x' = \frac{x'^2}{2}\ |_1^{1+x} = \frac{(1+x)^2-1}{2} = \mathbf{1}x + \mathbf{1}\frac{x^2}{2}$$

$$\int_1^{1+x} x'^2\,\mathrm{d}x' = \frac{x'^3}{3}\ |_1^{1+x} = \frac{(1+x)^3-1}{3} = \mathbf{1}x + \mathbf{2}\frac{x^2}{2} + \mathbf{1}\frac{x^3}{3}$$

$$\int_1^{1+x} x'^3\,\mathrm{d}x' = \frac{x'^4}{4}|_1^{1+x} = \frac{(1+x)^4-1}{4} = \mathbf{1}x + \mathbf{3}\frac{x^2}{2} + \mathbf{3}\frac{x^3}{3} + \mathbf{1}\frac{x^4}{4}$$

$$\int_1^{1+x} x'^4 dx' = \frac{x'^5}{5}\ |_1^{1+x} = \frac{(1+x)^5-1}{5} = \mathbf{1}x + \mathbf{4}\frac{x^2}{2} + \mathbf{6}\frac{x^3}{3} + \mathbf{4}\frac{x^4}{4} + \mathbf{1}\frac{x^5}{5}$$

$$\int_1^{1+x} x'^5 dx' = \frac{x'^5}{5}\ |_1^{1+x} = \frac{(1+x)^6-1}{6} = \mathbf{1}x + \mathbf{5}\frac{x^2}{2} + \mathbf{10}\frac{x^3}{3} + \mathbf{10}\frac{x^4}{4} + \mathbf{5}\frac{x^5}{5} + \mathbf{1}\frac{x^6}{6}$$

As in this table. In w^ch^ y^e^ first ar

x	X 1.	1.	1.	1.	1.	1.	1.	1.	1.	
$\frac{xx}{2}$	X −1.	0.	1.	2.	3.	4.	5.	6.	7.	
$\frac{x^3}{3}$	X 1.	0.	0.	1.	3.	6.	10.	15.	21.	
$\frac{x^4}{4}$	X −1.	0.	0.	0.	1.	4.	10.	20.	35.	
$\frac{x^5}{5}$	X 1.	0.	0.	0.	0.	1.	5.	15.	35.	
$\frac{x^6}{6}$	X −1.	0.	0.	0.	0.	0.	1.	6.	21.	
$\frac{1}{7}x^7$	X 1.	0.	0.	0.	0.	0.	0.	1.	7.	
		= abeb.	= adef.	= adeg.	= adeh.	&c				

de…

&…

Su… ab… &c.

it. By w^ch^ table it may appeare y^t^ y^e^ area of y^e^ Hyperbola adeb is $x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10}$ &c.